Carlos Alberto Corvini

FAST APPROACH SYSTEM (FAS) FOR LEARNING CALCULUS

SOLVING ORDINARY DIFFERENTIAL EQUATIONS

As a work philosophy FAS reflects scientific meticulousness, dedication and all the passion at the service of Mathematics

EDITORIAL DUNKEN

About This Book

FAST APPROACH SYSTEM (FAS) FOR LEARNING CALCULUS

Subject: Solving Ordinary Differential Equations

1. AUTHOR'S PROLOGUE

What is a prologue? One possible answer might be: a text seeking two intertwined objectives and which seem to go down different paths. On the one hand, introducing the word, previous step, preparation for speech. According to its etymology from the Greek $\pi\rho\acute{o}\lambda o\gamma o\varsigma$; *prologos*, from $\pi\rho\acute{o}$-***pro***: "before and towards"(in favor of), and $\lambda o\gamma o\varsigma$ ***logos***: "word". On the other hand, to argue not just as a viable logical way for a reasonable justification, but to persuade the receiver, implicit and model reader. The success in bringing these two parts together is what carries me to accept the challenge of writing this book.

The sole essence of the word seems to mark the way. ***Pro***: previous step towards the ***logos*** (argument). The ***pro*** may be interpreted as my experience, the fruit of my academic trajectory, of decades devoted to the teaching of Mathematics and constant and tenacious research to sustain this way of working. In this sense, the ***pro*** may be interpreted as the primary reason, the engine that starts the ***logos***, speech that endorses the **FAS (Fast Approach System) – my work philosophy.**

Although the attention of this book is focused on learning to solve **Ordinary Differential Equations (ODE)**, the philosophy sustaining it is applicable to learning any subject in Mathematics.

The learning of Mathematics poses a difficulty to more than one student. **FAS** helps to overcome these shortcomings or weaknesses, since it has been designed with an eminently practical and vital sense, with permanent concern for helping to think and stimulating creative thinking. Thus, Mathematics awakes passion due to its unfathomable object of study and for contributing new nuances to all sciences and to all of man's life since birth.

Returning to the concept of prologue, I may state that ***pro*** (preparation, experience) is the result of the research I have been perfecting since I was an advanced student majoring in Electrical Engineering at the ***Instituto Politécnico Superior*** in Rosario, Santa Fe, Argentina. During that period my book on "*Ejercicios sobre Ecuaciones Diferenciales*" was published, giving way to a transcendent effect on the way Mathematics was taught in the superior levels. I continued teaching differential equations to students of the different engineering branches, namely civil, mechanical, electronics and electrical at the ***Universidad Tecnológica Nacional***, and also Mathematical Analysis III at the *Facultad de Ciencias Exactas e Ingeniería* of the ***Universidad Nacional de Rosario***.

As founder and president of ***Instituto Superior Laplace de Ciencias en Sistemas*** (Rosario, Argentina) and through my professional career in universities abroad, I was able to prove that learning this work philosophy – **FAS**– based upon different approaches for learning the essential concepts yields permanent advances in students as regards Mathematics and study habits, helping them accomplish better academic performance. I, therefore, recognize **FAS** as an interesting pedagogical alternative, due to processes and results, for all the field of Mathematics.

In 2007 I attended ***California State University*** in Sacramento, California, USA, as a *visiting scholar*, invited by the Associate Dean of the *College of Natural Sciences & Mathematics*, Dr. Doraiswamy Ramachandran. Once there I exchanged classroom experiences with a group of faculty members from the *Math & Statistics Department*, among them Dr. David Zeigler and Dr. Andras Domokos, who regularly teach the courses of differential equations. Dr. David Zeigler, in his revision of this philosophy states: "***FAS*** *is a novel approach to teaching differential equations that places emphasis on the fundamental concepts and the interplay between various solution method*".

In 2012, **FAS** was first introduced in the *Mathematics Department* at ***Oregon State University*** in Corvallis, Oregon, USA. That same year it was also presented and sponsored by the *Maseeh Mathematics and Statistics Colloquium Series Fund* and the *Fariborz Maseeh Department of Mathematics and Statistics* at ***Portland State University*** in Oregon, USA. Today, **FAS** also appears in the treatment of online education.

If I expose in the concept of the prefix ***pro*** my career and experience, I now add to the root ***logos*** the argument that sustains the **FAS** philosophy. The objective in teaching Calculus is the understanding of concepts. In this work I propose a philosophy for learning a subject of Calculus, that is, a way of reaching the concepts from an equidistant equilibrium between a traditional orientation and a reformist one. Due to its structure, this system may be made as traditional and instructive (of the cookbook-recipe type) or as reformist (cutting out or taking different approaches) as one may want, thus stimulating logic and creative thinking.

The philosophical key in **FAS** is based upon the access to knowledge through a set of different approaches, each one of which has been created in the most careful way possible to achieve independence among them. Each one seeks to attain the maximum knowledge by itself. Graphically, we may imagine each approach as resting one next to the other, as in the tines of a fork. The idea consists in reaching the comprehension of the concepts through a ***strategy*** that *surfs* among the different approaches, consolidating the comprehension process and thus generating a way of thinking. **FAS** does not pretend to change the way of thinking incorporated in our unconscious but to connect to it so that together they may open the mind and bring out the best in everyone.

Due to the independence of each approach, each of them may be considered a book in itself. The new information that each approach provides is incorporated into the cognitive structure, connecting to preexisting knowledge. Each approach is not used as an end but as a means to cause a disintegration of the cognitive structure and its subsequent automatic restructuring.

Although the number of approaches is finite, the combinations of information processing are infinite, in the same manner as primary colors are blended into an infinite number of shades and hues. Summarizing, the more approaches we take, the more significant the learning, which will vary according to the intensity and order considered.

Going back to Dr. David Zeigler, he assures that "*because each approach is independent, the differential equations curriculum is flexible. An instructor may use a deliberate combination of the approaches to develop a curriculum that reflects the needs of the students. For example, a class composed primarily of engineers will see greater benefit from the methodological approach, while a class with a large percentage of mathematics majors will gain more understanding from the analytic approach. Furthermore, the decomposition of the material into the distinct approaches allows for a deeper investigation of the material without being distracted from the content appearing in other approaches.*"

We name the approaches ***direct*** and ***indirect***. The ***direct*** ones are called *analytic, methodologic* and *applications*, these being the main generators of knowledge. The ***indirect*** ones are two: *Reference Guide* and *Guided Practice*, which complement the system.

The reader has just started a new way of facing Science, life and the deep bond with human evolution.

Carlos A. Corvini

December 2018

2. OBJECTIVES AND BASIC CHARACTERISTICS (and raison d'être) OF THE FAST APPROACH SYSTEM

The system is addressed to:

- Professionals teaching Mathematics who need a practical and efficient pedagogical tool.
- Engineers, physicists and mathematicians who wish to refresh or expand their knowledge of ODE.
- High School and University students who may benefit through self-acquisition of tools and study habits, since the system provides flexibility and interaction similar to individual personalized education, thus obtaining better academic achievement and reducing eventual tutoring costs.
- Tutors teaching online classes due to the ductility of the system.

Among the benefits we may count the following:

- It encourages the three basic procedures to access knowledge and how to react properly: instinct, learning and comprehension.
- It allows students to promptly get to the subject of their interest without having to skip parts of the chapters in a book in which the subjects are not separate.
- It helps instructors who wish to present a minimum of each approach to do so without worrying about skipping parts of a chapter.

- It separates the different approaches to be able to pay special attention to each and every one of them.
- It reduces time and costs for education by live streaming and/or on demand.
- It reaches a vast number of students since it is a bilingual system (Spanish/English).
- It collaborates with teachers due to the fact that part of the material's structure may be replaced, generating a beneficial symbiosis in the teacher/student relationship, also providing a complement to curricula.

3. IMPORTANT OBSERVATION

This work has been focused on learning ordinary differential equations (ODE); in a set of pedagogical supports carefully designed to capture the readerŠs interest and increase comprehension when solving ODEs.

4. SYSTEM CONTENTS AND LAYOUT (for solving Ordinary Differential Equations)

The structure of the work material is divided into three main parts, which demand that readers interact with them regulating their progress at their own pace.

Part I	Reference Guide (indirect approach)
Part II	Analytic Approach (Chapter 1)
	Geometric Approach (Chapter 2)
	Methodologic Approach (Chapter 3)
	Applications (Chapter 4)
Part III	Guided Practice (indirect approach)

Part I: Reference Guide or Optimized Table

It contains the most important formulae, definitions and properties ordered in tables and arranged in a way that allows a quick reference and view of subject by subject, from Chapter 1 to Chapter 4. In other words, this guide represents the work of building the structural part of knowledge into a summary.

This Reference Guide will serve as an **initial guide** to ease learning and the comprehension of each of the approaches, and as a **final summary** for the Guided Practice, exams or to solve a proposed exercise.

Part II. Approaches

In this part we find the development of each approach to reach the maximum knowledge for solving ordinary differential equations (ODE). It contains the four chapters of the book.

Each approach has been worked upon in the most careful way to obtain independence from one another. Graphically, we may think of them as resting one alongside the other.

Starting with the analytic approach – or at least part of it – continuing with the approaches of our interest and interacting with the Reference Guide we shall obtain a particular knowledge.

Obviously, the more approaches we take, the vaster our knowledge will be. The path chosen will be known as the "***route of knowledge***".

> **Chapter 1. Analytic Approach.** It is the beginning and development of our first knowledge. It is dedicated to the development of procedures; that is, the implicit and explicit obtaining of formulae for solving different ordinary differential equations. In this approach we propose unsolved exercises so that readers may develop their intuition and discover the structure of the ODE by themselves.
>
> **Chapter 2. Geometric Approach.** It deals with the geometric procedures for representing ODE solutions.
>
> **Chapter 3. Methodologic Approach.** "More systematic methods" are developed, such as the Operational Method, a mathematical technique that has become a powerful instrument in ODE solving. Also, the Numerical Method which is used to calculate estimated values of unknown functions, for certain values of the independent variable. Sometimes, for many of the difficult ODEs, these are the only practical approaches.
>
> **Chapter 4. Applications Approach.** The concepts of ODE solving are presented through application problems. We propose different problems that frequently come up in Physics and Engineering for readers to formulate in terms of an ODE; that is, to build a mathematical model associated to the problem and solve it.
> However, given the reciprocity between the ODEs and their applications, readers may

develop their intuition and discover by themselves the corresponding ODE structure. And by exercising problem solving also attain operative skills.

Part III: Guided Practice

It consists of a set of exercises and partially solved problems without consideration for degrees of difficulty.

Although this Guided Practice does not directly represent an approach in itself, it is good for consolidating concepts and increasing knowledge. This part may be considered as an indirect approach. However, we must point out that this part is the most important one for it boosts the comprehension process and emphasizes a “regulated” way of thinking.

5. CONCLUSION

As time moves on there is more need for tools to acquire knowledge. **FAS** is one of them. This asseveration could probably be criticized, but it is an opinion formed in the evidence of experiences acquired throughout time.

From a pedagogical point of view, **FAS** represents another alternative for understanding concepts and in no way intends to replace others. Technological tools are complementary and may be used for teaching online, face-to-face or both simultaneously.

If to understand a concept, we start from a traditional approach or simplified vertical teaching process, we have a definition, then practice, a geometric interpretation, a numerical calculation and also applications or application problems, ending up with an evaluation. This path offers us a minimum result. Whereas, if from each part of this sequence we try to generate independent approaches to reach the understanding of a concept, chances of arriving at an optimum result increase.

The independence of the different approaches speaks of intensity of study of each one of them. And the interaction among them allows the student to “regulate” his progress and his studies at his own pace. For this reason, we may speak of “regulated thinking”.

The effort is focused on “improving” rationally and mathematically regulated work towards the understanding of the concept. Improving is increasing the amount of approaches or independent alternatives and the interaction among them in a horizontal movement.

When we encounter an unknown mathematical exercise or problem, it is not just about finding an algorithm to solve the problem, since we would be facing an exercise. For **FAS** when face to face with a problem whose solution is not reached immediately, its heuristic method consists of breaking it up into known fragments and solving it using knowledge acquired with the different approaches of the route of knowledge chosen, combining it with our own knowledge acquired before, and then taking a path to the final solution. **FAS** considers the solution of a problem not just as a situation that requires a solution, but also as a consolidation of concepts and solving methods of each fragment. Thus, the importance of the applications or problem-solving approach.

As a final conclusion, I may state that **FAS**, as a work philosophy, reflects scientific meticulousness, dedication and all the passion at the service of education, reason for which I hope this material turns out useful for all those who delve into these pages.

Rosario, Santa Fe, Argentina

December 2018

TABLE OF CONTENTS

Page

REFERENCE GUIDE

SOLVING DIFFERENTIAL EQUATIONS

1. Separable Variables

$P_1(x)Q_1(y)dx + P_2(x)Q_2(y)dy = 0$

Variables x and y can be separated by the equals sign.

Solution: $\displaystyle\int \frac{P_1(x)}{P_2(x)}dx + \int \frac{Q_2(y)}{Q_1(y)}dy = C$

2. Homogeneous

$\dfrac{dy}{dx} = \psi\left(\dfrac{y}{x}\right)$

Solution: $\dfrac{e^{\int \frac{dv}{\psi(v)-v}}}{x} = C \quad \text{where} \quad \dfrac{y}{x} = v$

Observation: If $\psi(v) = v$ the solution is $y = Cx$.

3. Exact

$P(x,y)dx + Q(x,y)dy = 0$

where $\quad \dfrac{\partial P(x,y)}{\partial y} = \dfrac{\partial Q(x,y)}{\partial x}$

Solution: $\int P(x,y)dx + \int \left(Q(x,y) - \frac{\partial}{\partial y}\int P(x,y)dx\right) dy = C$

dx indicates the integral is with respect to x, being y constant.

4. Non Exact

$P(x,y)dx + Q(x,y)dy = 0$

where $\quad \dfrac{\partial P(x,y)}{\partial y} \neq \dfrac{\partial Q(x,y)}{\partial x}$.

Solution: **Method of the Integrating Factor.** Multiply all terms by

$\mu = e^{\int \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)dx}$ or $\mu = e^{-\int \frac{1}{P}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)dy}$

and it is transformed into exact. μ is called ***integrating factor***.

5. Reducible to Separable Variables

$\dfrac{dy}{dx} = f(ax + by + c)$

Variables not directly separable.

Solution: $\displaystyle\int \frac{du}{a + b + f(u)} = x + C$ where $u = u(x) = ax + by + c$

6. Reducible to Homogeneous or Separable Variables

$\dfrac{dy}{dx} = f\left(\dfrac{ay + bx + c}{a_1 y + b_1 x + c_1}\right)$

c or c_1 non zero.

a) $a_1 b - b_1 a \neq 0$

Solution: It is reduced to a homogeneous equation $\dfrac{d\eta}{d\xi} = f\left(\dfrac{a\left(\frac{\eta}{\xi}\right) + b}{a_1\left(\frac{\eta}{\xi}\right) + b_1}\right)$

where $\xi = x - x_0$ and $\eta = y - y_0$.

(x_0, y_0) point of intersection of the lines $ay + bx + c = 0$ and $a_1 y + b_1 x + c_1 = 0$

b) $a_1 b - b_1 a = 0$

Solution: It is reduced to one of separable variables $v' = af\left(\dfrac{v}{\lambda v - \lambda c + c_1}\right) + b$

where

$v = ay + bx + c$

$v' = ay' + b$

$a_1 = \lambda a$

$b_1 = \lambda b$

Observation: If $c = c_1 = 0$, it is computed directly as a homogeneous.

7. First Order Linear Equation

$\dfrac{dy}{dx} + P(x)y = Q(x)$

Solution: $y = e^{-\int P(x)dx}\left[\int Q(x)e^{\int P(x)dx}dx + C\right]$

8. The Bernoulli Equation

a) Order 1 and degree n

$\dfrac{dy}{dx} + P(x)y = Q(x)y^n$

$n \neq 1$

Solution: $y = e^{-\int P(x)dx}\left[(1-n)\int Q(x)e^{(1-n)\int P(x)dx} + C\right]^{\frac{1}{1-n}}$

b) Order 1 and degree 1. $\dfrac{dy}{dx} + P(x)y = Q(x)y$

Solution: $y = Ce^{\int (Q(x)-P(x))dx}$

9. Ricatti of order 1 and degree 2

$\dfrac{dy}{dx} = P(x)y^2 + Q(x)y + W(x)$

$y = y_1(x)$ particular solution, that is $y_1^{'}(x) = Py_1^2(x) + Qy_1(x) + W$

$y'(x) = y'_1(x) - \dfrac{u'(x)}{u^2(x)}$

Solution: It is reduced to a first order linear equation.

$u' + (2y_1(x)P(x) + Q(x))\,u + P(x) = 0$

Substituting $y(x) - y_1(x) = \dfrac{1}{u(x)}$

$y'(x) = y_1(x) - \dfrac{u(x)}{u^2(x)}$

Observation: If $P(x) = 0$, item 9 is transformed into one of type 7.

10. Second Order, Incomplete

a) $\dfrac{d^2y}{dx^2} = F(x)$

Solution: $y = \iint F(x)dx^2 + C_1x + C_2$

b) $\dfrac{d^2y}{dx^2} + P(x)\dfrac{dy}{dx} + Q(x) = 0$

Solution: $y = \int \left[e^{-\int P(x)dx}\left(-\int Q(x)e^{\int P(x)dx}dx + C_1\right)\right] dx + C_2$

from making $p = \dfrac{dy}{dx}$ and reducing to a linear equation of first order

$$\frac{dp}{dx} + P(x)p + Q(x) = 0$$

c) $\dfrac{d^2y}{dx^2} + P(x)\dfrac{dy}{dx} = Q(x)\left(\dfrac{dy}{dx}\right)^n$

Solution: It is reduced to one of the Bernoulli type $\dfrac{dp}{dx} + P(x)p = Qp^n$

substituting $p = \dfrac{dy}{dx}$

d) $f(y, y', y'') = 0$

Solution: It is reduced to one of first order through substitution

$$\frac{dy}{dx} = p \quad \text{then} \quad \frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dy}{dx}\frac{dp}{dy} = p\frac{dp}{dy}$$

11. Homogeneous Linear Equation of Second Order

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + qy = 0$$

p and q are real-valued constants.

Solution:

Let the auxiliary equation be $r^2 + pr + q = 0$ and the roots r_1, r_2.

Case 1: $r_1 \neq r_2$ real

$$y = C_1e^{r_1x} + C_2e^{r_2x}$$

Case 2: $r_1 = r_2 = r$ real

$$y = C_1e^{rx} + C_2xe^{rx}$$

Case 3: $r_1 = a + bi \quad r_2 = a - bi$

$y = e^{ax}\left(C_1 \cos bx + C_2 \sin bx\right)$

Observation: In all cases y receives the name of complementary function.

12. Non Homogeneous Linear Equation of Second Order

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = f(x)$$

where p and q are real-valued constants.

Solution: $y = y_{\text{complementary}} + y_{\text{particular integral}}$

The $y_{\text{complementary}}$ corresponds to the three cases in Item 11.

The $y_{\text{particular integral}}$ depends on $f(x)$.

Case 1:

$$y = C_1 e^{r_1 x} + C_2 e^{r_2 x} + \frac{e^{r_1 x}}{r_1 - r_2}\int e^{-r_1 x} f(x)dx + \frac{e^{r_2 x}}{r_2 - r_1}\int e^{-r_2 x} f(x)dx$$

Case 2:

$$y = C_1 e^{rx} + C_2 x e^{rx} + x e^{rx}\int e^{-rx} f(x)dx - e^{rx}\int x e^{-rx} f(x)dx$$

Case 3:

$$y = e^{ax}\left(C_1 \cos bx + C_2 \sin bx\right) + \tfrac{e^{ax}\sin bx}{b}\int e^{-ax} f(x)\cos bx dx - \tfrac{e^{ax}\cos bx}{b}\int e^{-ax} f(x)\sin bx dx$$

13. Homogeneous Linear Equation of Order n

Form: $\dfrac{d^n y}{dx^n} + p_1\dfrac{d^{n-1}y}{dx^{n-1}} + p_2\dfrac{d^{n-2}y}{dx^{n-1}} + p_3\dfrac{d^{n-3}y}{dx^{n-3}} + ... + p_n y = 0$

where $p_1, p_2, ..., p_n$ are real-valued constants.

Solution: Let the auxiliary equation be $r^n + p_1 r^{n-1} + ... + p_n = 0$ and the roots $r_1, ..., r_n$.

Case 1: $r_1, r_2, ..., r_n$ are real-valued and different.

$$y = C_1 e^{r_1 x} + C_2 e^{r_2 x} + ... + C_n e^{r_n x}$$

Case 2: We have simple multiple roots.

Pages 6 to 35 are not included in this book preview.

The series converges for $|x| < 1$ where (22) represents a particular integral.

b) **The Hypergeometric Series**

We define the hypergeometric series as the expression within brackets in (20). It can be written as

$$F(\alpha, \beta, \gamma, x) = 1 + \frac{\alpha\beta}{1\gamma} + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1 \times 2\gamma(\gamma+1)}x^2 + \dots$$

c) **Various Properties**

i) (21) can be written as $y = a_0 F(\alpha, \beta, \gamma, x)$

ii) $F\left(n+1, -n, 1, \dfrac{1-x}{2}\right) = P_n(x)$

iii) $\dfrac{d}{dx}F(\alpha, \beta, \gamma, x) = \dfrac{\alpha\beta}{\gamma}F(\alpha+1, \beta+1, \gamma+1, x)$

iv) $F(\alpha, \beta, \beta x) = \dfrac{1}{(1-x)^{\alpha}}$

v) $F(1, \beta, \beta x) = 1 + x + x^2 + x^3 + ... + x^n = \dfrac{1}{1-x}$

vi) $F(1, 1, 2, x) = \dfrac{(1-x)}{x}$

d) **General Solution (The Frobenius Method)**

The solution is given by the series $y = x^C \sum_{k=0}^{\infty} a_k x^k$ where C is a coefficient to be determined and gives us the recurrence equation $a_{n+1} = \dfrac{(n+C+\alpha)(n+C+\beta)}{(n+C+1)(n+C+\gamma)}a_n \quad n \geq 0$

(The Frobenius Method).

We obtain for $C = 0$ the expression in (21), and for $C = 1 - \gamma$

$$a_{n+1} = \frac{(n+1-\gamma+\alpha)(n+1-\gamma+\beta)}{(n+2-\gamma)(n+1)}a_n \quad n \geq 0$$

Then, expanding and setting the constant to $a_0 = b_0$, to make it different from the one before,

$$y_2 = b_0 x^{1-\gamma}\left[1 + \sum_{n=1}^{\infty} \frac{(n-\gamma+\alpha)\dots(1-\gamma+\alpha)(n-\gamma+\beta)\dots(1-\gamma+\beta)}{(n+1-\gamma)\dots(4-\alpha)(3-\alpha)(2-\gamma)n!} x_n\right]$$

Then $y_6 = y_1 + y_2 = a_0 F(\alpha,\beta,\gamma,x) + b_0 x^{1-\gamma} F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma, x)$

4) THE BESSEL DIFFERENTIAL EQUATION

a) Solution:

$$x^2 y'' + xy' + (x^2 - n^2)y = 0 \tag{23}$$

By the series $y = x^C \sum_{k=0}^{\infty} a_k x^k$ (C constant to be determined), the recurrence equation is

$$a_{2k} = \frac{(-1)^k a_0}{2^{2k} k!(n+1)(n+2)...(n+k)} \quad k = 1, 2, 3, \dots$$

And let $a_0 = \dfrac{1}{2^n \Gamma(n+1)}$ where $\Gamma(k+1)$ is the gamma function. We obtain the particular solutions which we will call Bessel functions of the first kind and order n.

$$J_n(x) = \frac{x^n}{2^n \Gamma(n+1)} \left\{1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2 \times 4(2n+2)(2n+4)} - \dots\right\} =$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(n+k+1)} \left(\frac{x}{2}\right)^{n+2k}$$

$$J_{-n}(x) = \frac{x^{-n}}{2^{-n}\Gamma(1-n)} \left\{1 - \frac{x^2}{2(2-2n)} + \frac{x^4}{2 \times 4(2-2n)} - \dots\right\} =$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+1-n)} \left(\frac{x}{2}\right)^{2k-n}$$

Observation: If n is a positive integer or zero $\Gamma(n+k+1) = (n+k)!$

i) $J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!}\left(\frac{x}{2}\right)^{n+2k}$

ii) $J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2} - \frac{x^6}{2^2 4^2 6^2} + \ldots$

iii) $J_1(x) = \frac{x}{2} - \frac{x^3}{2^2 4} + \frac{x^5}{2^2 4^2 6} - \frac{x^7}{2^2 4^2 6^2 8}$

iv) $J_0(x) = -J_1(x)$

v) $J_n(0) = \begin{cases} 0 & n > 0 \\ 1 & n = 0 \end{cases}$

vi) $J_{-n}(x) = (-1)^n J_n(x) \quad n = 0, 1, 2, 3, \ldots$

vii) $J_n(x)$ and $J_{-n}(x)$ are independent if $n \neq 0, 1, 2, 3$

b) Bessel functions of the second kind and order n

Let n be non integer. We define the Bessel function of the second kind and order n as the function given by

$$Y_n(x) = \frac{J_n(x)\cos n\pi - J_{-n}(x)}{\sin n\pi} \qquad n \neq 0, 1, 2, 3, 4$$

$$Y_n = \lim_{P \to n} Y_P(x) \quad n = 0, 1, 2$$

c) General solution of the Bessel differential equation.

i) $y = C_1 J_n(x) + BJ_{-n}(x) \quad n \neq 0, 1, 2, ...$

ii) $y = C_1 J_n(x) + BY_n(x)$ for every n

5) **The Thomson Functions or Ber-bei Functions**

Let the equation $x\frac{d^2y}{dx^2} + \frac{dy}{dx} - ixy = 0$

Its solution is given by $y = J_0(i^{\frac{3}{2}}x)$. Substituting in the expansion of $J_0(x)$

$$J_0(i^{\frac{3}{2}}x) = \left[1 + \sum_{k=1}^{\infty} \frac{(-1)^k x^{4k}}{2^2 4^2 6^2 8^2 (4k)^2}\right] + i\left[-\sum_{k=1}^{\infty} (-1)^k \frac{x^{4k-2}}{2^2 4^2 6^2 8^2 ...(4k-2)^2}\right]$$

Naming the brackets

$$J_0(i^{\frac{3}{2}}x) = ber(x) + i\ bei(x)$$

This page intentionally left blank

Chapter 1

ANALYTIC APPROACH

1 Introduction

A ***differential equation*** is that in which we find derivatives or differentials of any order of functions that have to be determined. Differential equations are divided into ***ordinary***, for example:

$$\frac{3x+y}{x+8y} = \frac{dy}{dx} \tag{1.1}$$

$$\frac{d^2y}{dx^2} + 3\frac{d^2y}{dx^2} + y = 0 \tag{1.2}$$

in which the function to be determined depends on a single variable.

And into ***partial derivatives***, as:

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0 \tag{1.3}$$

in which the function z to be determined depends on two variables.

By "order" of a differential equation we mean the derivative of highest order appearing in said equation. For example,

$$\frac{dy}{dx} + 3y = x^2 + x \tag{1.4}$$

This equation is of *first order* because the highest appearing order is 1. Instead,

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0 \tag{1.5}$$

is of *second order* because the maximum appearing order is 2.

If the derivative of highest order is n, for example:

$$(n \geq 1) \qquad \frac{d^n y}{dx^n} + \frac{d^{n-1} y}{dx^{n-1}} + \ldots + y = 0 \tag{1.6}$$

we say that *the differential equation is of nth order.*

By *degree* of a differential equation we mean the power to which the derivatives are raised. For instance,

$$\frac{d^2 y}{dx^2} + \frac{dy}{dx} + y = 0 \tag{1.7}$$

is a differential equation of second order and first degree.

Instead,

$$\left(\frac{dy}{dx}\right)^2 = 3x + 2 \tag{1.8}$$

is a differential equation of second degree.

If we have two or more dependent variables under a same independent variable we use systems of "simultaneous" differential equations to obtain the value of each of the dependent variables. For example,

$$\begin{cases} \dfrac{dy}{dt} = x + y \\ \dfrac{dx}{dt} = 3x + 1 \end{cases} \tag{1.9}$$

where x and y are the dependent variables and t is the independent variable.

For our study, we will present differential equations in different forms, showing the derivatives as:

$$\frac{dy}{dx} \;\; ; \;\; y' \;\; ; \;\; \mathrm{D}(y) = \frac{dy}{dx} \tag{1.10}$$

2 Solving differential equations

Let's assume a given differential equation

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0 \tag{2.1}$$

which, in this case, represents one of second order.

$$c_1 = \text{ constant (arbitrary)} \qquad y = c_1 e^{2x} \tag{2.2}$$

is an integral solution to (2.1) since

$$\frac{dy}{dx} = 2c_1 e^{2x} \tag{2.3}$$

$$\frac{d^2y}{dx^2} = 4c_1 e^{2x} \tag{2.4}$$

and replacing (2.2), (2.3), (2.4) in (2.1), satisfies the equation

$$4c_1 e^{2x} - 10c_1 e^{2x} + 6c_1 e^{2x} = 0 \tag{2.5}$$

Thus, we observe that $y = c_1 e^{2x}$ is a solution to (2.1).

In the same manner, we can verify that

$$c_2 = \text{ constant (arbitrary)} \qquad y = c_2 e^{3x} \tag{2.6}$$

is another solution to (2.1).

If we establish

$$y = c_1 e^{2x} + c_2 e^{3x} \tag{2.7}$$

we can see that it also satisfies (2.1).

As can be seen, by assigning arbitrary values to c_1 and c_2 we will obtain infinite solutions to the given differential equation.

$y = c_1 e^{2x}$ and $y = c_2 e^{3x}$ are called *particular solutions.* Instead, $y = c_1 e^{2x} + c_2 e^{3x}$ is called general solution because it is a complete solution to (2.1).

Below we will see different ways of solving differential equations and the various forms in which they appear.

We will begin our study with ***variables separable***.

3 Equations with separable variables

When we have a differential equation of the form

$$\frac{dy}{dx} = f(x, y) \tag{3.1}$$

it may be solved through separation of variables, when it can be written in the form

$$P(x)dx + Q(y)dy = 0 \tag{3.2}$$

Then, integrating the equation, we have

$$\int P(x)dx + \int Q(y)dy = C \tag{3.3}$$

where C represents an arbitrary constant.

Thus we obtain the general solution to the equation (3.1).

Below are some examples to illustrate this kind of differential equations.

***Example* 3.1.** Let's assume we have the differential equation

$$\frac{dy}{dx} = yx^2 \tag{a}$$

This equation can be reduced to the form (3.2) as

$$\frac{dy}{y} - x^2dx = 0 \tag{b}$$

Then, integrating we have

$$\int \frac{dy}{y} - \int x^2dx = c \tag{c}$$

$$\ln\, y - \frac{x^3}{3} = c \tag{d}$$

Simplifying

$$y = e^{\frac{x^3}{3}+c} = Ce^{\frac{x^3}{3}} \qquad C = e^c \tag{e}$$

That is, (e) satisfies (a), and is a general solution.

***Example* 3.2.** Solve the following differential equation

$$\frac{dy}{dx} = \frac{3x - y}{x + 2y} \tag{a}$$

Clearing fractions (a) is transformed into

$$xdy + ydx + 2y\, dy - 3x\, dx = 0 \tag{b}$$

Looking at (b) we deduct that $xdy + ydx = d(xy)$ and then, by integrating we have

$$xy + y^2 - \frac{3}{2}x^2 = C \tag{c}$$

which is a general solution to the differential equation (a).

***Example* 3.3.** Solve the following differential equation

$$\frac{dy}{dx} = \frac{y - 2x^3y}{x + x^4} \tag{a}$$

Clearing denominators

$$xdy - ydx + x^4dy + 2x^3ydx = 0 \tag{b}$$

Dividing by x^2 we have

$$\frac{xdy - ydx}{x^2} + x^2dy + 2xydx = 0 \tag{c}$$

Since

$$d\left(\frac{y}{x}\right) = \frac{xdy - ydx}{x^2} \qquad \text{and} \qquad d(x^2y) = x^2dy + 2xydx$$

Integrating

$$\frac{y}{x} + x^2y = C \tag{d}$$

which is the general solution to (a).

Exercises

Solve the following differential equations:

		SOLUTION
1.	$\dfrac{dy}{dx} = \dfrac{y}{x}$	$\dfrac{y}{x} = C$
2.	$\dfrac{dy}{dx} = yx^2$	$y = Ce^{\frac{x^3}{3}}$
3.	$\dfrac{dy}{dx} = \dfrac{y}{x-3}$	$y = (x-3)C$
4.	$\dfrac{dy}{dx} = \dfrac{x}{2+y}$	$4y + y^2 - x^2 = C$
5.	$\dfrac{dy}{dx} = \dfrac{e^x}{e^y+1}$	$e^y + y - e^x = C$
6.	$\dfrac{dy}{dx} = \sec y \sin x$	$\sin y + \cos x = C$
7.	$\dfrac{dy}{dx} = \dfrac{-y-\cos x}{x}$	$xy + \sin x = C$
8.	$\dfrac{dy}{dx} = -\dfrac{y}{x-y}$	$2xy - y^2 = C$
9.	$\dfrac{dy}{dx} = -\dfrac{y}{x+x^2}$	$\dfrac{xy+y}{x} = C$
10.	$\dfrac{dy}{dx} = \dfrac{y}{x+x^2\cos y}$	$\dfrac{y}{x} + \sin y = C$
11.	$\dfrac{dy}{dx} = \dfrac{\sin x - y}{x}$	$\cos y + xy = C$
12.	$\dfrac{dy}{dx} = -\dfrac{xy}{x^2+1}$	$y^2(x^2+1) = C$
13.	$\dfrac{dy}{dx} = \dfrac{y(-2x-1)}{x(x+1)}$	$x^2y + xy = C$
14.	$\dfrac{dy}{dx} = -\dfrac{ye^x+y+e^x}{xe^x+x}$	$xy + \ln(e^x+1) = C$
15.	$\dfrac{dy}{dx} = \dfrac{y-2x^2\cos x}{x}$	$\dfrac{y}{x} + 2\sin x = C$

Pages 47 to 143 are not included in this book preview.

24 Variation of constants method

If we have the differential equation of the form

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \tag{24.1}$$

where P, Q, R are functions of x or constants.

The general solution is given by the sum of the complementary solution and the particular integral, that is:

$$y_g = y_c + y_p \tag{24.2}$$

where the complementary solution is given as

$$y_c = c_1y_1 + c_2y_2 \tag{24.3}$$

If we want to find the particular integral of the form $y_p = C_1y_1 + C_2y_2$ through variation of constants, thus obtaining the system

$$\begin{cases} C_1^{'}y_1 + C_2^{'}y_2 = 0 \\ C_1^{'}y_1^{'} + C_2^{'}y_2^{'} = R \end{cases} \tag{24.4}$$

where C_1 and C_2 are functions of x.

Eliminating said functions from the system in (24.4) yields

$$\begin{cases} C_1^{'}(y_1y_2^{'} - y_1^{'}y_2) = -y_2R_2 \\ C_2^{'}(y_1y_2^{'} - y_1^{'}y_2) = y_1R \end{cases} \tag{24.5}$$

and since

$$y_1y_2^{'} - y_1^{'}y_2 \neq 0 \tag{24.6}$$

we have

$$C_1 = -\int \frac{y_2R}{y_1y_2^{'} - y_1^{'}y_2}dx \ ; \ C_2 = \int \frac{y_1R}{y_1y_2^{'} - y_1^{'}y_2}dx \tag{24.7}$$

from this

$$y_p = -y_1 \int \frac{y_2 R}{y_1 y_2^{'} - y_1^{'} y_2} dx + y_2 \int \frac{y_1 R}{y_1 y_2^{'} - y_1^{'} y_2} dx \tag{24.8}$$

Let us solve an example that will illustrate the case.

***Example* 24.1.** Solve the following differential equation.

$$\frac{d^2y}{dx^2} + \frac{x}{x^2+1}\frac{dy}{dx} - \frac{y}{x^2+1} = x^2 + 1 \tag{a}$$

Equating this equation to zero we obtain the complementary solution through the method studied in Item 21 that is,

$$\frac{d^2y}{dx^2} + \frac{x}{x^2+1}\frac{dy}{dx} - \frac{y}{x^2+1} = 0 \tag{b}$$

Since u is worth x, the complementary solution is given by

$$y_c = c_1 x + c_2 x \int \frac{e^{-\int \frac{x}{x^2+1} dx}}{x^2} dx = c_1 x + c_2 x \int \frac{dx}{x^2\sqrt{x^2+1}} \tag{c}$$

integrating

$$y_c = c_1 x + c_2 \sqrt{x^2+1} \tag{d}$$

Here

$$\begin{aligned} y_1 &= x \qquad & y_2 &= \sqrt{x^2+1} \\ y_1^{'} &= 1 \qquad & y_2^{'} &= \tfrac{x}{\sqrt{x^2+1}} \end{aligned} \tag{e}$$

$$y_1 y_2^{'} - y_1^{'} y_2 = -\frac{1}{\sqrt{x^2+1}} \tag{f}$$

Applying the formula in (24.7)

$$C_1 = \int \frac{\sqrt{x^2+1}(x^2+1)}{\sqrt{x^2+1}} dx = \frac{x^3}{3} + x$$

$$C_2 = -\int \frac{x(x^2+1)}{\sqrt{x^2+1}} dx = \int x\sqrt{x^2+1} dx = -\frac{1}{3}(x^2+1)^{\frac{3}{2}}$$

From (24.8)

$$y_p = \frac{x^4}{3} + x^2 - \frac{(x^2+1)^2}{3}$$

$$y_p = \frac{x^2}{3} - \frac{1}{3}$$

Then, the general solution is

$$y = c_1 x + c_2\sqrt{x^2+1} + \frac{x^2}{3} - \frac{1}{3}$$

25 THE ADJOINT AND SELF-ADJOINT EQUATIONS

A) ADJOINT EQUATIONS

a) If we have a differential equation of the form

$$f_2\frac{d^2y}{dx^2} + f_1\frac{dy}{dx} + f_0 y = f \tag{25.1}$$

where f may be zero or non zero.

We said it was exact if we could verify that

$$f_2^{''} - f_1^{'} + f_0 = 0 \tag{25.2}$$

In the case that (25.1) is not exact, i.e. we cannot verify (25.2), let us attempt to find a factor to make (25.1) exact. If we multiply (25.1) by u

$$uf_2\frac{d^2y}{dx^2} + uf_1\frac{dy}{dx} + uf_0 y = uf \tag{25.3}$$

For (25.3) to be exact we must have

$$(uf_2)'' - (uf_1)' + uf_0 = 0 \tag{25.4}$$

Differentiating

$$u''f_2 + 2u'f_2^{'} + uf_2^{''} - u'f_1 - uf_1^{'} + uf_0 = 0 \tag{25.5}$$

Collecting terms

$$f_2u'' + (2f_2' - f_1)u' + (f_2'' - f_1' + f_0)u = 0 \tag{25.6}$$

The equation in (25.6) is called ***adjoint equation*** of (25.1). Once we find factor u we substitute it in the equation in (25.3), making it exact. Then, we solve using Item 20.

***Example* 25.1.** Solve the following differential equation.

$$x^2\frac{d^2y}{dx^2} + 5x\frac{dy}{dx} + 4y = 3x + 2 \tag{a}$$

Applying (25.6)

$$x^2u'' + (4x - 5x)u' + (2 - 5 + 4)u = 0 \tag{b}$$

or

$$x^2u'' - xu' + u = 0 \tag{c}$$

From (c) we have

$$u = x \tag{d}$$

Multiplying the equation in (a) by (d) we have

$$x^3\frac{d^2y}{dx^2} + 5x^2\frac{dy}{dx} + 4xy = 3x^2 + 2x \tag{e}$$

As can be seen, we can verify the expression

$$f_0 - f_2' + f_2'' = 0 \tag{f}$$

That is, $4x - 10x + 6x = 0$, which implies that (e) is exact. Applying Item 20 we have

$$x^3y' + 2x^2y = x^3 + x^2 + C_1 \tag{g}$$

Then

$$y = \frac{1}{x^2}\left[\int \left(x^2 + x + \frac{C_1}{x}\right)dx + C_2\right]$$

$$y = \frac{x}{3} + \frac{1}{2} + \frac{C_1 \ln x}{x^2} + \frac{C_2}{x}$$

which represents the general solution to (25.1).

B) **SELF–ADJOINT EQUATIONS**

If we have a differential equation of the type in (25.1); that is,

$$f_2 \frac{d^2y}{dx^2} + f_1 \frac{dy}{dx} + f_0 y = f \tag{25.7}$$

and we verify that

$$f_2' = f_1 \tag{25.8}$$

the equation in (25.6) becomes

$$f_2 u'' + (2f_1 - f_1)u' + (f_1' - f_1' + f_0)u = 0 \tag{25.9}$$

or

$$f_2 u'' + f_1 u' + f_0 u = 0$$

That is, the adjoint equation in (25.1) is the same equation in (25.1) equated to zero, provided that we can verify the identity in (25.8).

The differential equations in (25.7) are called ***self–adjoint equations***.

***Example* 25.2.** Solve the following differential equation:

$$\frac{1}{x}\frac{d^2y}{dx^2} - \frac{1}{x^2}\frac{dy}{dx} + \frac{y}{x^3} = 0 \tag{a}$$

Here

$$f_2 = \frac{1}{x} \qquad f_2' = -\frac{1}{x^2} \qquad f_1 = -\frac{1}{x^2} \tag{b}$$

As can be seen, we can verify (25.8). This means the adjoint equation coincides with the equation in (a).

$$\frac{1}{x}u'' - \frac{1}{x^2}u' + \frac{u}{x^3} = 0 \tag{c}$$

From there

$$u = x \tag{d}$$

Then,

$$\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} + \frac{y}{x^2} = 0 \tag{e}$$

Once we multiply the equation in (a) by u, it becomes the exact one in (e) since we can verify

$$f_0 - f_2' + f_1'' = 0 \tag{f}$$

Due to this

$$\frac{dy}{dx} - \frac{y}{x} = C_1 \tag{g}$$

Computing

$$y = \int \frac{C_1}{x} dx + C_2 \tag{h}$$

$$y = C_1 x \ln x + C_2 x \tag{i}$$

being this the general solution to (a).

A differential equation of the form

$$f_2 \frac{d^2y}{dx^2} + f_1 \frac{dy}{dx} + f_0 y = f \tag{25.10}$$

can be transformed into the self-adjoint form and can be written as

$$\frac{d^2y}{dx^2} + \frac{f_1}{f_2}\frac{dy}{dx} + \frac{f_0}{f_2} = \frac{f}{f_2} \tag{25.11}$$

The self-adjoint form of an equation can be written as

$$A\frac{d^2y}{dx^2} + A'\frac{dy}{dx} + By = C \tag{25.12}$$

or

$$\frac{d^2y}{dx^2} + \frac{A'}{A}\frac{dy}{dx} + \frac{B}{A}y = \frac{C}{A} \tag{25.13}$$

From (25.13)

$$\frac{A'}{A} = \frac{d\ln A}{dx} \tag{25.14}$$

And since (25.14) has to be equal to $\dfrac{f_1}{f_2}$

$$\frac{d\ln A}{dx} = \frac{f_1}{f_2} \Rightarrow A = e^{\int \frac{f_1}{f_2} dx} \tag{25.15}$$

Then

$$\frac{B}{A} = \frac{f_0}{f_2} \Rightarrow B = \frac{f_0}{f_2} e^{\int \frac{f_1}{f_2} dx} \tag{25.16}$$

$$\frac{C}{A} = \frac{f}{f_2} \Rightarrow C = \frac{f}{f_2} e^{\int \frac{f_1}{f_2} dx} \tag{25.17}$$

Substituting values in (25.15), (25.16) and (25.17) in (25.12) gives the equation in (25.10) in its self-adjoint form.

$$e^{\int \frac{f_1}{f_2} dx} \frac{d^2y}{dx^2} + \frac{f_1}{f_2} e^{\int \frac{f_1}{f_2} dx} \frac{dy}{dx} + \frac{f_0}{f_2} e^{\int \frac{f_1}{f_2} dx} y = \frac{f}{f_2} e^{\int \frac{f_1}{f_2} dx} \tag{25.18}$$

As we can observe from (25.18), to transform a differential equation (25.10) into an self-adjoint we must multiply the equation by the value of z:

$$z = \frac{1}{f_2} e^{\int \frac{f_1}{f_2} dx} \tag{25.19}$$

Observation 25.1. *In the latter expression* (25.19)*, if f_1 is the derivative of f_2, z takes the value of 1; that is, the equation in* (25.1) *is an self-adjoint.*

***Example* 25.3.** Solve the following differential equation by transforming it first into an self-adjoint.

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = x^2 \tag{a}$$

Multiplying this equation by

$$z = \frac{1}{f_2} e^{\int \frac{f_1}{f_2} dx} = e^x$$

since in this case $f_1 = 1$, $f_2 = 1$, that is

$$e^x \frac{d^2y}{dx^2} + e^x \frac{dy}{dx} - 2e^x y = x^2 e^x \tag{b}$$

The latter is the self-adjoint form of (a) and it is not exact. Applying (25.6) we obtain

$$e^x u'' + e^x u' - 2e^x u = 0 \tag{c}$$

From here $u = e^x$.

Then, the expression in (b) becomes exact when we multiply it by u. That is,

$$e^{2x} \frac{d^2y}{dx^2} + e^{2x} \frac{dy}{dx} - 2e^{2x} y = x^2 e^{2x} \tag{d}$$

Since (d) is exact, applying the expression in (20.9) gives

$$e^{2x}y' + (-2e^{2x} + e^{2x})y = \int x^2 e^{2x} dx + C_1 \tag{e}$$

or

$$e^{2x}y' + e^{2x}y = \frac{e^{2x}}{2}\left(x^2 - x + \frac{1}{2}\right) + C_1 \tag{f}$$

$$y' - y = \frac{x^2}{2} - \frac{x}{2} + \frac{1}{4} + \frac{C_1}{e^{2x}} \tag{g}$$

Then

$$y = e^x\left[\int\left(\frac{x^2}{2} - \frac{x}{2} + \frac{1}{4} + \frac{C_1}{e^{2x}}\right)e^{-x}dx + C_2\right] \tag{h}$$

$$y = e^x\left[\int \frac{x^2}{2}e^{-x}dx - \int \frac{xe^{-x}}{2}dx + \int \frac{e^{-x}}{4}dx + \int \frac{C_1}{e^{3x}}dx + C_2\right] \tag{i}$$

Integrating

$$y = \frac{x^2}{2} - \frac{x}{2} - \frac{3}{4} + C_1 e^{-2x} + C_2 e^x \tag{j}$$

which represents the general solution to (a).

Exercises

Solve the following exercises using the methods presented in this Item.

		SOLUTION
1)	$x^3\dfrac{d^2y}{dx^2} + 4x^2\dfrac{dy}{dx} + 2xy = 0$	$y = \dfrac{C_1}{x} - \dfrac{C_2}{x^2}$
2)	$\dfrac{d^2y}{dx^2} + \dfrac{1}{x}\dfrac{dy}{dx} - \dfrac{y}{x^2} = 2x$	$y = \dfrac{x^3}{4} + \dfrac{C_1 x}{2} + \dfrac{C_2}{x}$
3)	$(1-x)\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} = 1 - 2x$	$y = \dfrac{x^2}{2} - C_1\ln(1-x) + C_2$
4)	$(x^2+1)\dfrac{d^2y}{dx^2} + 4x\dfrac{dy}{dx} + 2y = 0$	$y = \dfrac{xC_1 + C_2}{x^2+1}$

Page 152 is not included in this book preview.

Chapter 2

GEOMETRIC APPROACH

1 Geometric interpretation of differential equations

If we have a first order differential equation in the form

$$y' = f(x, y) \tag{1.1}$$

with the solution

$$F(x, y, C) = 0 \tag{1.2}$$

or

$$y = \varphi(x, C) \tag{1.3}$$

As may be seen in the latter expression, (1.3) represents a family of solutions, by assigning different values to the constant C.

If we plot equation (1.3) in the cartesian plane, assigning C an arbitrary value, we obtain an integral curve of the differential equation in (1.1).

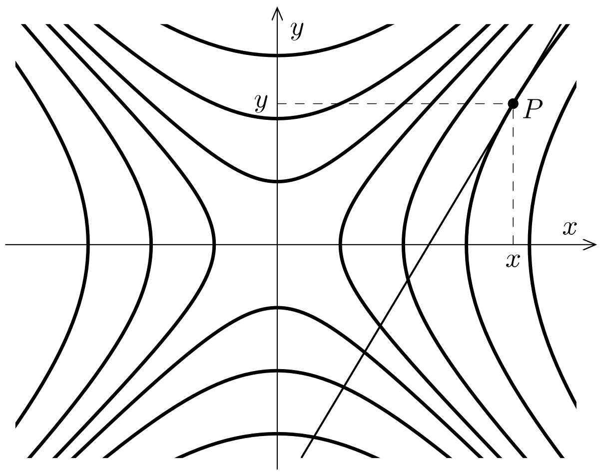

Figure 2.1

If constant C takes all values we obtain infinite solutions that are represented by infinite curves, which are called ***family of integral curves***. (Figure 2.1)

A first order first degree equation determines a single slope in any point $P(x, y)$ and thus a single integral curve going through that point.

***Example* 1.1.** Find the equation of the curve that goes through point $P(2, 1)$ and satisfies the differential equation:

$$y' + x = 0 \tag{a}$$

Solving through separation of variables we obtain the equation of the infinite integral curves, that is

$$y = -\frac{x^2}{2} + C \tag{b}$$

which represents a family of parabolas. (Figure 2.2)

Since the curves belonging to this family must go through point $P(2, 1)$, in (b) we substitute the values of the given point for x and y, thus obtaining

$$1 = -\frac{4}{2} + C \tag{c}$$

where C is worth 3. Substituting this value in (b) we have the equation of the curve going through the given point which satisfies the differential equation in (a). That is,

$$y = -\frac{x^2}{2} + 3 \tag{d}$$

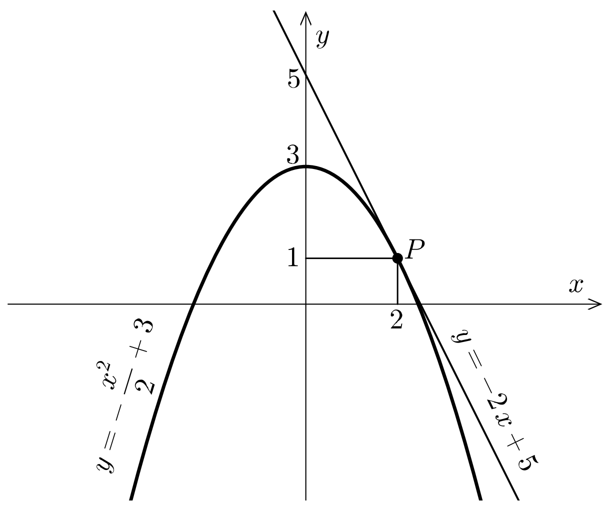

Figure 2.2

Substituting 2 for x in (a) we obtain the slope of the line tangent to curve $y = -\dfrac{x^2}{2} + 3$ on point $P(2, 1)$. This line's equation is $y = -2x + 5$.

2 Method of the Isoclines

Many times it is convenient to apply methods to obtain the solutions of differential equations that would be difficult to solve in an analytical manner.

One of these methods is that of the ***isoclines***; a graphical method that is very convenient when the solution is not so hard to plot.

Let's consider the following differential equation:

$$y' = f(x, y) \tag{2.1}$$

By assigning values to x and y we obtain a series of slopes, which represent those of the tangent lines to the integral curves. Representing the different points with their corresponding slopes on the cartesian plane gives us a field of directions (Figure 2.3), where the segments correspond to tangent lines to the different integral curves. This is called ***directional field***.

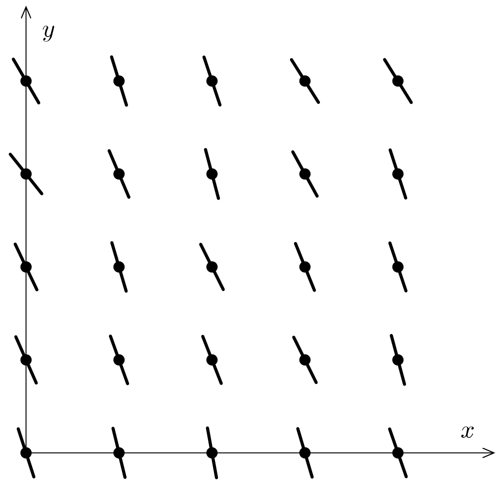

Figure 2.3

Representing integral curves with points of the directional field is very complicated. We reduce the work by joining all the points of the segments with like slope.

That is, in (2.1)

$$y' = K \qquad K \text{ constant} \tag{2.2}$$

Or the same

$$f(x, y) = K \tag{2.3}$$

This equation which satisfies curves going through points of like slope are called ***isocline curves***. (Figure 2.4)

As we know, integral curves must intercept the isoclines tangent to the segments that each of the latter intercept. That is, to plot integral curves satisfying the differential equation in (2.1) we must proceed in the following manner: We assign different values to constant K in the expression in (2.3), $K_1, K_2, K_3, \ldots$. If we take value K_1 we have

$$y_1 = F(x, K_1) \tag{2.4}$$

This is a curve representing an isocline which cuts segments of K_1 slope.

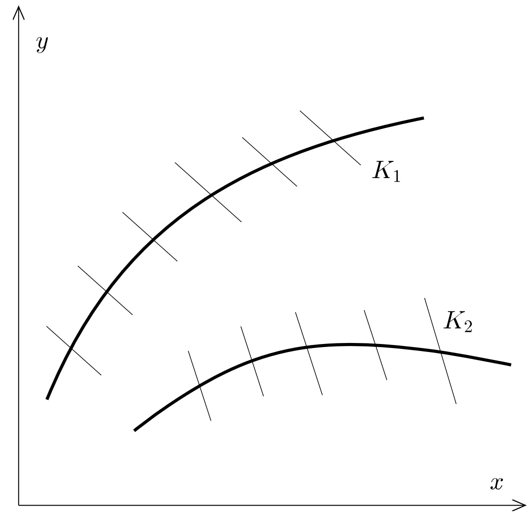

Figure 2.4

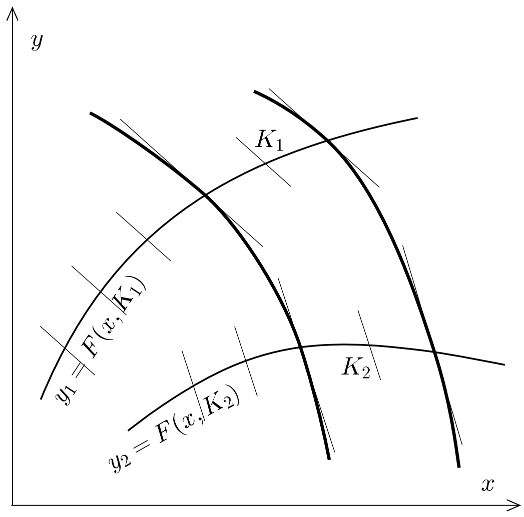

Figure 2.5

Repeating the process and varying the value of K as many times as exactitude requires, we obtain the family of isoclines. Then, we may draw the integral curves, tangent to the different segments, as can be observed in Figure 2.5.

This method of the isoclines has the advantage of giving us all of the integral curves.

***Example* 2.1.** Find the integral curves satisfying the following differential equation:

$$y' = 2x \tag{a}$$

By making

$$K = 2x \tag{b}$$

we thus obtain the isocline curves given by the equation

$$x = \frac{K}{2} \tag{c}$$

which represent parallel lines to the y axis, giving different values to constant K. (Figure 2.6)

That is, if in the expression in (c) we make

$$K = 0 \quad \text{we have} \quad x = 0$$

$$K = \frac{1}{2} \qquad\qquad x = \frac{1}{4}$$

$K = 1$	$x = \dfrac{1}{2}$
$K = 2$	$x = 1$
$K = 4$	$x = 2$
$K = 6$	$x = 3$
$K = -\dfrac{1}{2}$	$x = -\dfrac{1}{4}$
$K = -1$	$x = -\dfrac{1}{2}$
$K = -2$	$x = -1$
$K = -4$	$x = -2$
$K = -6$	$x = -3$

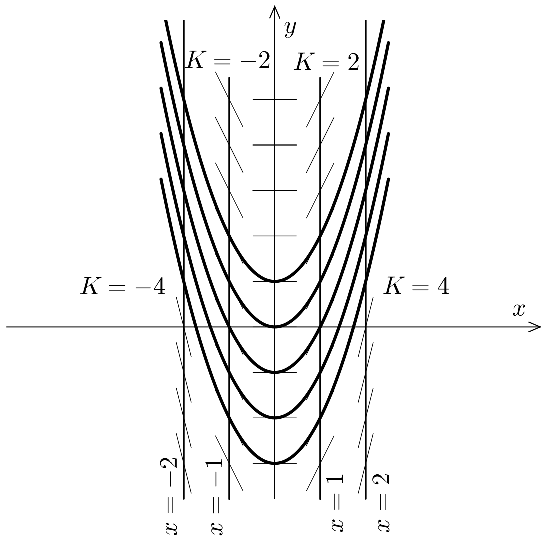

Figure 2.6

***Example* 2.2.** Find the integral curves that satisfy the following differential equation

$$y' = \frac{y}{2x} \tag{a}$$

By making

$$K = \frac{y}{2x} \tag{b}$$

That is

$$y = 2Kx \tag{c}$$

we thus obtain the equation of the isoclines that represent lines going through the origin. If in the expression in (c) we make

$K = -3$	we have	$y = -6x$
$K = -2$		$y = -4x$
$K = -1$		$y = -2x$

$K = -\dfrac{1}{2} \qquad y = -x$

$K = 0 \qquad y = 0$

$K = \dfrac{1}{2} \qquad y = x$

$K = 1 \qquad y = 2x$

$K = 2 \qquad y = 4x$

$K = 3 \qquad y = 6x$

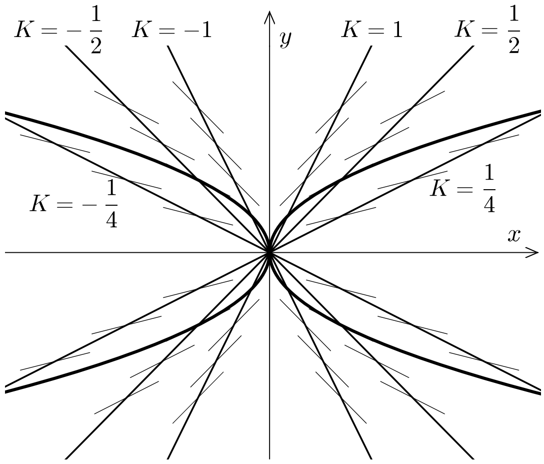

Figure 2.7

These values allow us to plot the integral curves satisfying the differential equation in (a). These curves represent parabolas having the equation $y = c\sqrt{x}$. (Figure 2.7)

***Example* 2.3.** Find the integral curves satisfying the differential equation

$$y' = \frac{2x}{2y - 3x} \tag{a}$$

By making

$$K = \frac{2x}{2y - 3x} \tag{b}$$

where

$$y = \left(\frac{2 + 3K}{2K}\right) x = mx \tag{c}$$

This is the equation of the isoclines representing lines going through the origin.

If we vary the value of K, that is

$K = 0 \qquad$ we have $\qquad m = \infty$

$K = \infty \qquad m = \dfrac{3}{2}$

$K = -\dfrac{2}{3} \qquad m = 0$

$K = 1 \qquad m = 2.5$

$$K = -1 \qquad m = \frac{1}{2}$$

$$K = -\frac{1}{3} \qquad m = -\frac{3}{2}$$

We obtain a series of directions with which we can plot the integral curves that satisfy the differential equation in (a). (Figure 2.8)

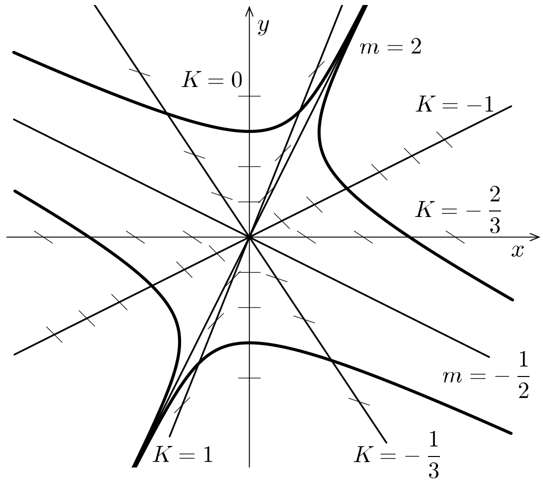

Figure 2.8

The asymptotes may be obtained in the following manner:

Since the value of K must be equal to that of m, from (c) we have

$$K = \frac{2}{2m - 3} \tag{d}$$

By making the right side equal to m:

$$\frac{2}{2m - 3} = m \tag{e}$$

we thus obtain a second degree equation

$$m^2 - \frac{3}{2}m - 1 = 0$$

Solving we obtain

$$m_1 = 2 \qquad m_2 = -\frac{1}{2}$$

These values represent the slopes of both asymptotes, which turn out to be perpendicular to each other.

***Example* 2.4.** Find the integral curves satisfying the differential equation.

$$y' = \frac{y - x}{y} \tag{a}$$

We make

$$K = \frac{y - x}{y} \tag{b}$$

where

$$y = \left(\frac{1}{1-K}\right) x = mx \tag{c}$$

is the equation for the isoclines representing the lines that go through the origin.

If in (c) we make

$K = 1$ we have $m = \infty$

$K = 0$ $\quad m = 1$

$K = \infty$ $\quad m = 0$

$K = 2$ $\quad m = -1$

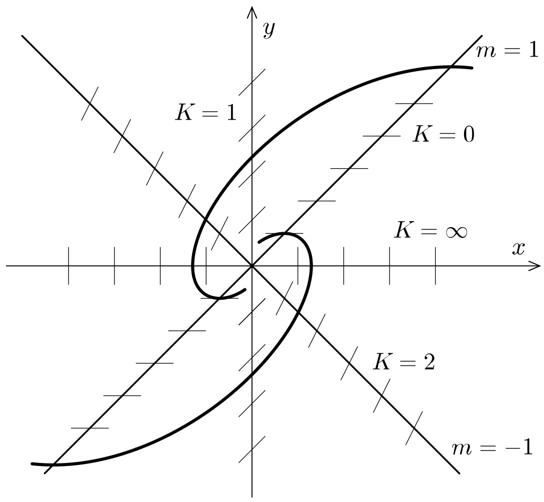

Figure 2.9

These values allow us to plot the integral curves. (Figure 2.9)

If we make $K = m$ we obtain a second degree equation yielding solutions with complex roots, reason for which there are no asymptotes.

***Example* 2.5.** Find the integral curves satisfying the differential equation.

$$y' = x^2 + y^2 \tag{a}$$

where the equation for the isoclines is

$$x^2 + y^2 = K \tag{b}$$

which represent concentric circles.

By making

$K = 1$ we have $x^2 + y^2 = 1$

$K = 4$ $\quad x^2 + y^2 = 2^2$

$K = 9$ $\quad x^2 + y^2 = 3^2$

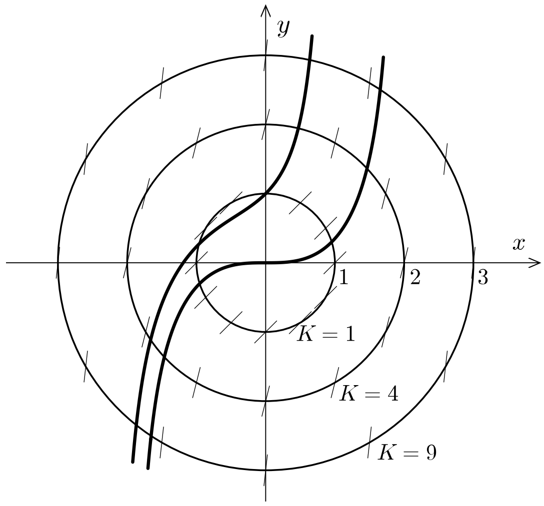

Figure 2.10

These values allow us to plot the integral curves. (Figure 2.10)

3 Geometric problems

***Example* 3.1.**

a) Find the differential equation whose solution represents the ∞ parabolas.

$$y = ax^2 \tag{a}$$

Since by assigning different values to a we obtain infinite curves, if we eliminate the constant by differentiating in (a) we obtain a differential equation, which expresses a property common to all curves.

That is, differentiating in (a) we have

$$y' = 2ax \tag{b}$$

from where

$$a = \frac{y'}{2x} \tag{c}$$

Substituting (c) in (a) we have

$$2y = y'x \tag{d}$$

The solution to this differential equation represents parabolas whose vortexes coincide with the origin. (Figure 2.11)

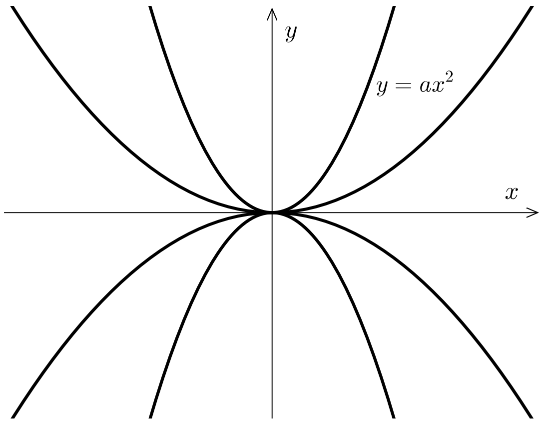

Figure 2.11

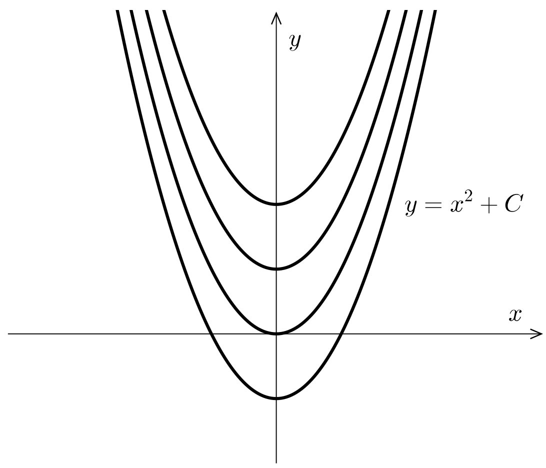

Figure 2.12

Pages 162 to 186 are not included in this book preview.

11 The Euler-Cauchy Method

a) Let the following be a first order differential equation of the form

$$y' = f(x, y) \tag{11.1}$$

We know that if we assign a pair of values (x_i, y_i) to the solution of (11.1), we obtain the value of constant C of the integral curve going through that point. Given a point $P_0(x_0, y_0)$, the Euler-Cauchy method consists of finding an integral curve going through that point.

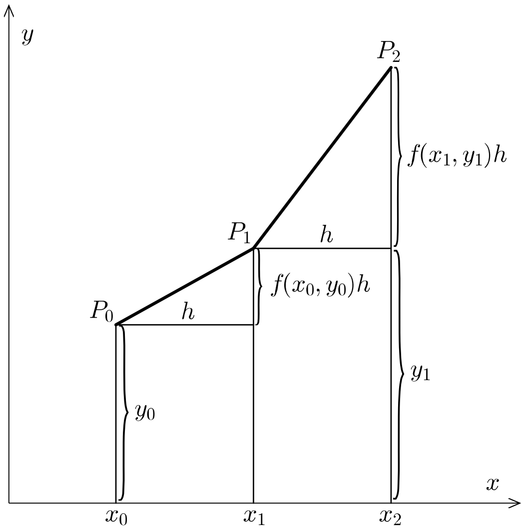

Figure 2.35

As we can see in Figure 2.35, translating the tangent to the integral curve from point P_0 to point P_1 we obtain the coordinates of the latter point, which are:

$$\begin{aligned} x_1 &= x_0 + h \\ y_1 &= y_0 + \Delta y_0 = y_0 + f(x_0, y_0)h \end{aligned} \tag{11.2}$$

Then, we use this pair of values to compute the new slope $f(x_1, y_1)$, obtaining an approximated value. With this slope and the pair of values (x_1, y_1) we obtain the new approximate values of

$$\begin{aligned} &x_2 = x_1 + h && y_2 = y_1 + \Delta y_1 = y_1 + f(x_1, y_1)h \\ &x_3 = x_2 + h && y_3 = y_2 + \Delta y_2 = y_2 + f(x_2, y_2)h \\ &\quad\vdots && \quad\vdots \\ &x_n = x_{(n-1)} + h && y_n = y_{(n-1)} + \Delta y_{(n-1)} = y_{(n-1)} + f(x_{(n-1)}, y_{(n-1)})h \end{aligned} \tag{11.3}$$

We so obtain a series of values (x_i, y_i), which determine an approximate solution in the form of a polygonal called ***Euler-Cauchy polygonal***.

This polygonal is less likely as we move farther from given point P_0. The error that appears when computing y_1 is dragged on to compute slope y_2. That is, the values y_n beyond $n = 1$ are affected by errors, which become greater as the equation becomes more dependent on y. In other words, this method is not advisable for functions that vary fast with the argument. Thus, we must take the smallest possible values of h to decrease error.

***Example* 11.1.** Find the values of y for $x = 1.1, 1.2, 1.3, 1.4, \ldots, 1.9, 2.0$ knowing that $y_0 = 1$ for $x_0 = 1$, which satisfy the differential equation

$$y' = 2\frac{y}{x} \tag{a}$$

Applying what we saw in (11.2) and (11.3)

$$x_0 = 1 \qquad y_0 = 1$$

$$x_1 = 1 + h = 1 + 0.1 = 1.1 \qquad y_1 = 1 + \Delta y_0 = 1 + f(x_0, y_0)h = 1 + \frac{2.1}{1}0.1 = 1.2$$

$$x_2 = 1.1 + 0.1 = 1.2 \qquad y_2 = 1.2 + 2\frac{1.2}{1.1}0.1 \cong 1.418$$

And proceeding in this manner we obtain a table of values:

x	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
y	1	1.20	1.42	1.65	1.91	2.18	2.47	2.78	3.10	3.45	3.81
y'	2	2.18	2.36	2.54	2.72	2.90	3.08	3.26	3.44	3.62	3.81
$y = x^2$	1	1.21	1.44	1.69	1.96	2.25	2.56	2.89	3.24	3.61	4
Error	–	0.01	0.02	0.04	0.05	0.07	0.09	0.11	0.14	0.16	0.19

The values were approximated up to the second digit after the decimal point.

The latter two lines in the table show the exact value for y and the error committed by the method. We can see that as we move away from point $P(1, 1)$ the error increases.

b) Up to this point we have studied the Euler-Cauchy method in relation to first order differential equations. We will now deal with a method for solving second order differential equations of the form:

$$y'' = f(x, y, y') \tag{11.4}$$

The Euler-Cauchy method provided us with a solution in the form of a polygonal. For a second order differential equation as the one in (11.4) we see that on a point x_0 we obtain an ordinate y_0 and a slope y_0' (Figure 2.36).

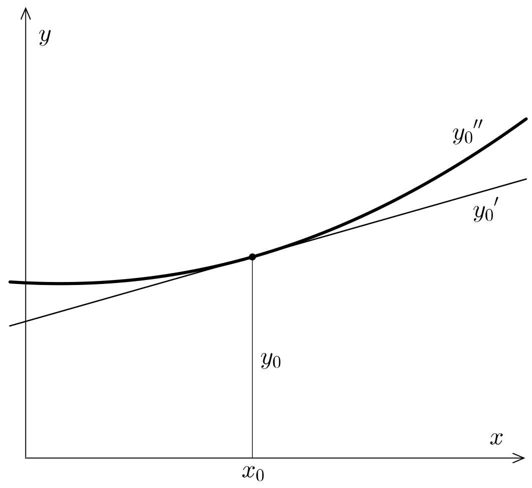

Figure 2.36

The latter values imply a concavity y_0'' that is constant up to a point $x_1 = x_0 + h$.

This curve is a parabola of the form:

$$y = y_0 + y_0'x + y''\frac{x^2}{2} \tag{11.5}$$

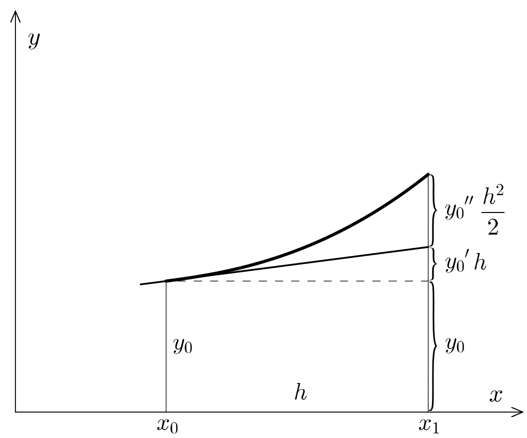

Figure 2.37

Then, from (11.5), for a very small interval h we will have a value y_1. (Figure 2.37)

$$y_1 = y_0 + y_0'h + y_0''\frac{h^2}{2} \tag{11.6}$$

Differentiating (11.6) we have

$$y_1' = y_0' + y_0''h \tag{11.7}$$

And substituting in (11.4) gives

$$y_1'' = f(x_1, y_1, y_1') \tag{11.8}$$

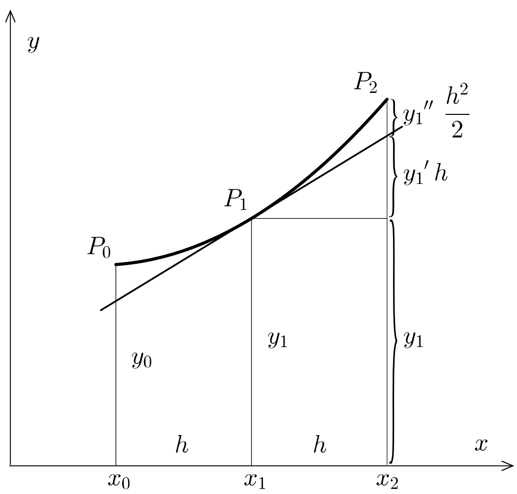

Figure 2.38

That is, a new concavity y_1'', value which allows us to compute a new value for y. (Figure 2.38)

$$y_2 = y_1 + y_1'h + y''\frac{h^2}{2} \tag{11.9}$$

This concavity y_1'' determines a new parabola arc that goes from x_1 to $x_1 + h = x_2$.

And successively we obtain the different values for y_n, which determine a solution in the form of a parabolic polygonal, with a constant concavity from point x_i to $x_i + h$.

Using the Taylor series and making $x - x_0 = h$

$$y = y_0 + y_0' h + y_0'' \frac{h^2}{2!} + y_0''' \frac{h^3}{3!} + \dots \tag{11.10}$$

This method coincides up to the third term, thus implying that the error is of order h^3. Instead, in the equations of first order we had an error in the order of h^2, greater because the value of y coincided up to the second term.

***Example* 11.2.** Solve the following differential equation:

$$y'' = 2y - 2x^2 \tag{a}$$

with initial conditions $y_0 = 1$ and $y_0' = 0$ when $x_0 = 0$.

For $x = 0,\ 0.2,\ 0.4,\ 0.6,\ 0.8$ and $h = 0.2$

Solution: $y = x^2 + 1$

x_i	y_i	$y_i' h$	$y_i'' \frac{h^2}{2} = f_i \frac{h^2}{2} = k_i$	y **exact**	**Error**
$x_0 = 0$	$y_0 = 1$	$y_0' h = 0 \times 0.2 = 0$	$k_0 = 2\frac{0.2^2}{2} = 0.04$	1	–
$x_1 = x_0 + h$ $x_1 = 0.2$	$y_1 = y_0 + y_0' h + y_0'' \frac{h^2}{2}$ $y_1 = 1 + 0 + 0.04 = 1.04$ $y_1 = 1.04$	$y_1' h = y_0' h + 2k_0$ $y_1' h = 0 + 0.08 = 0.08$	$k_1 = 0.04$	1.04	–
$x_2 = x_1 + h$ $x_2 = 0.4$	$y_2 = y_1 + y_1' h + y_1'' \frac{h^2}{2}$ $y_2 = 1.04 + 0.08 + 0.04 = 1.16$ $y_2 = 1.04$	$y_2' h = y_1' h + 2k_1$ $y_2' h = 0.08 + 0.08 = 0.16$	$k_2 = 0.04$	1.16	–
$x_3 = x_2 + h$ $x_3 = 0.6$	$y_3 = y_2 + y_2' h + y_2'' \frac{h^2}{2}$ $y_3 = 1.16 + 0.16 + 0.04 = 1.36$ $y_3 = 1.36$	$y_3' h = y_2' h + 2k_2$ $y_3' h = 0.16 + 0.08 = 0.24$	$k_3 = 0.04$	1.36	–
$x_4 = x_3 + h$ $x_4 = 0.8$	$y_4 = y_3 + y_3' h + y_3'' \frac{h^2}{2}$ $y_4 = 1.36 + 0.24 + 0.04 = 1.64$ $y_4 = 1.64$	$y_4' h = y_3' h + 2k_3$ $y_4' h = 0.24 + 0.08 = 0.32$	$k_4 = 0.04$	1.64	–
$x_5 = x_4 + h$ $x_5 = 1$	$y_5 = 1.64 + 0.32 + 0.04 = 2$ $y_5 = 2$	$y_5' h = y_4' h + 2k_4$ $y_5' h = 0.32 + 0.08 = 0.4$	$k_4 = 0.04$	2	–
$x_6 = x_5 + h$	$y_6 = 2 + 0.4 + 0.04 = 2.44$ $y_6 = 2.44$	$y_5' h = y_4' h + 2k_4$	–	–	–

Note: Values $f_i\dfrac{h^2}{2} = k_i = y_i''\dfrac{h^2}{2}$ were found in the following manner:

For $x = 0$ and $y = 1$:

$$y_0'' = 2 = f_0 \quad \text{then} \quad k_0 = 2\frac{0.2^2}{2} = 0.04$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

and so on successively to find the k_i values.

As we can observe there were no errors. This does not mean that the method is exact, but that the solution to the differential equation is a second degree parabola, which signifies that the arc concavities are all constant.

In a general case, the parabolic polygonal would increasingly drift from the real solution as we got farther away from point x_0.

This page intentionally left blank

Chapter 3

METHODOLOGIC APPROACH

Operational and Numerical Methods for Solving Differential Equations

Introduction

We will divide this chapter into three parts, namely:

- *Part One*: It deals with *operator* D as a method for solving linear differential equations with constant coefficients. Although, in the analytic approach, we have already seen how to solve these equations by the variation of constants and undetermined coefficients, *operator* D helps us shorten the way to finding general solutions. Besides, *operator* D allows us to find a particular integral that satisfies the initial or boundary conditions without having to first find the general solution, thus being a good tool for engineering applications.

- *Part Two*: It deals with the Laplace Transform as a mathematical instrument for solving differential equations. It is worthy noting the use of this method in solving linear differential equations with constant coefficients, where the independent term is a function of the periodic or impulsive type. We will also observe the *relation* of Laplace Transforms with the solving of linear differential equations with variable coefficients, such as polynomials of the independent variable, because of their importance in mathematical applications of different fields of Physics and Engineering. (See *Applications Approach*)

- *Part Three*: This is an introduction to methods for numerical solving of differential equations, considering the series as a base of those methods.

Part I Integration of linear differential equations using operator D

1 Operator "D"

Operator D is defined as $\frac{d}{dx} \equiv \mathrm{D}$, and acts on a function $y = f(x)$ as

$$\mathrm{D}\, f(x) = \mathrm{D}\, y = \frac{d}{dx} y = \frac{dy}{dx} = y' \tag{1.1}$$

Then, applying D to (1.1)

$$\mathrm{D}\,\mathrm{D}\, f(x) = \mathrm{D}\,\mathrm{D}\, y = \frac{d}{dx}\left(\frac{d}{dx} y\right) = \frac{d^2 y}{dx^2} = \frac{dy'}{dx} = y'' \tag{1.2}$$

We convene that $\mathrm{D}\,\mathrm{D} = \mathrm{D}^2$, and repeating the procedure,

$$\mathrm{D}^n\, y = \frac{d^n y}{dx^n} = y^n$$

Because of the former we have the following properties:

I. $\mathrm{D}^n (y_1 + y_2) = \frac{d^n (y_1 + y_2)}{dx^n} = \frac{d^n y_1}{dx^n} + \frac{d^n y_2}{dx^n} = \mathrm{D}^n\, y_1 + \mathrm{D}^n\, y_2,$

II. $\mathrm{D}^n (py) = \frac{d^n (py)}{dx^n} = p\frac{d^n y}{dx^n} = p\,\mathrm{D}^n\, y$ with p constant,

III. $\mathrm{D}^m (\mathrm{D}^n\, y) = \frac{d^m}{dx^m}\left(\frac{d^n}{dx^n} y\right) = \frac{d^{m+n} y}{dx^{m+n}} = \mathrm{D}^{m+n}\, y$

IV. $(\mathrm{D}^m + \mathrm{D}^n) y = (\mathrm{D}^n + \mathrm{D}^m) y$

V. $[\mathrm{D}^m + (\mathrm{D}^n + \mathrm{D}^s)]\, y = [(\mathrm{D}^m + \mathrm{D}^n) + \mathrm{D}^s]\, y$

VI. $(\mathrm{D}^m\, \mathrm{D}^n) y = (\mathrm{D}^n\, \mathrm{D}^m) y$

VII. $(\mathrm{D}-\alpha)(\mathrm{D}-\beta)y = (\mathrm{D}-\alpha)(\mathrm{D}\,y-\beta y) = \mathrm{D}^2\,y-(\beta+\alpha)\,\mathrm{D}\,y+\alpha\beta y = (\mathrm{D}-\alpha)(\mathrm{D}-\beta)y$ with α, β constants.

Property VII tells us that D commutes for translation by constants α and β. Nonetheless, we must bear in mind that this is not always possible if α and β are not constants. As a consequence of the aforementioned properties, operator D is linear. Besides, it has the properties of power and factors just like polynomials.

In other words, operators behave as if they were algebraic quantities and can be manipulated algebraically subject to the laws above.

Operating

$$\mathrm{D}^n\,y + p_1\,\mathrm{D}^{n-1}\,y + \ldots + p_n y = (\mathrm{D}^n + p_1\,\mathrm{D}^{n-1} + p_2\,\mathrm{D}^{n-2} + \ldots + p_n)y \tag{1.3}$$

with $p_1, \ldots, p_n$ constants. We then convene it can be written in brief as $\Phi(\mathrm{D})y$.

We will now proceed to consider properties and formulas that contain operators.

Let $\mathrm{D}\,e^{rx} = re^{rx}$; $\mathrm{D}^2\,e^{rx} = r^2e^{rx}$; $\ldots$; $\mathrm{D}^n\,e^{rx} = r^ne^{rx}$.

Then

$$\begin{aligned}\Phi(\mathrm{D})e^{rx} &= (\mathrm{D}^n + p_1\,\mathrm{D}^{n-1} + p_2\,\mathrm{D}^{n-2} + \ldots + p_{n-1}\,\mathrm{D} + p_n)e^{rx} = \\ &= (r^n + p_1r^{n-1} + p_2r^{n-2} + \ldots + p_{n-1}r + p_n)e^{rx} = \Phi(r)e^{rx}\end{aligned} \tag{1.4}$$

And $\Phi(\mathrm{D})e^{rx} = \Phi(r)e^{rx}$.

Now let $y = e^{rx}\varphi(x)$

$$\mathrm{D}(e^{rx}\varphi(x)) = e^{rx}\,\mathrm{D}\,\varphi(x) + re^{rx}\varphi(x) = e^{rx}(\mathrm{D}+r)\varphi(x)$$

$$\begin{aligned}\mathrm{D}^2(e^{rx}\varphi(x)) &= \mathrm{D}\left[e^{rx}(\mathrm{D}+r)\varphi(x)\right] = e^{rx}(\mathrm{D}^2 + r\,\mathrm{D})\varphi(x) + re^{rx}(\mathrm{D}+r)\varphi(x) \\ &= e^{rx}(\mathrm{D}^2 + 2r\,\mathrm{D} + r^2)\varphi(x) = e^{rx}(\mathrm{D}+r)^2\varphi(x).\end{aligned}$$

Repeating the procedure above we may suppose the following expression valid for all k and by induction see for $k+1$.

$$\mathrm{D}^k(e^{rx}\varphi(x)) = e^{rx}(\mathrm{D}+r)^k\varphi(x). \tag{1.5}$$

Let's see for k+1. Operating both members of (1.5), then

$$\begin{aligned}\mathrm{D}^{k+1}(e^{rx}\varphi(x)) &= \mathrm{D}\left[e^{rx}(\mathrm{D}+r)^k\varphi(x)\right] = e^{rx}\,\mathrm{D}(\mathrm{D}+r)^k\varphi(x) + re^{rx}(\mathrm{D}+r)^k\varphi(x) = \\ &= e^{rx}(\mathrm{D}+r)(\mathrm{D}+r)^k\varphi(x) = e^{rx}(\mathrm{D}+r)^{k+1}\varphi(x)\end{aligned}$$

Then it is valid for $k+1$. That is, (1.5) is valid for all k.

Let us now see applying the validity of (1.5)

$$\begin{aligned}\Phi(\mathrm{D})y &= \Phi(\mathrm{D})(e^{rx}\varphi(x)) = \\ &= (\mathrm{D}^n + p_1\,\mathrm{D}^{n-1} + \ldots + p_{n-1}\,\mathrm{D} + p_n)(e^{rx}\varphi(x)) = \\ &= e^{rx}\left[(\mathrm{D}+r)^n + p_1\,(\mathrm{D}+r)^{n-1} + \ldots + p_{n-1}\,(\mathrm{D}+r) + p_n\right]\varphi(x) = \\ &= e^{rx}\Phi(\mathrm{D}+r)\varphi(x).\end{aligned} \tag{1.6}$$

If in property (1.6) we make $\Phi(\mathrm{D}) = (\mathrm{D}-r)^k$

$$(\mathrm{D}-r)^k\,(e^{rx}\varphi(x)) = e^{rx}\,\mathrm{D}^k\,\varphi(x) \tag{1.7}$$

And considering that $y = e^{rx}\varphi(x)$, equation (1.7) may be written as

$$(\mathrm{D}-r)^k\,y = e^{rx}\,\mathrm{D}^k(e^{-rx}y) \tag{1.8}$$

With these elements we will deal with differential equations through the application of operators.

2 INTEGRATION OF HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS

Let the homogeneous linear differential equation with constant coefficients be

$$\frac{d^ny}{dx^n} + p_1\frac{d^{n-1}y}{dx^{n-1}} + p_2\frac{d^{n-2}y}{dx^{n-2}} + \ldots + p_{n-1}\frac{dy}{dx} + p_ny = 0 \tag{2.1}$$

$$p_1, \ldots, p_n \text{ constants.}$$

Through the change of notations seen before:

$$(\mathrm{D}^n + p_1 \mathrm{D}^{n-1} + \ldots + p_{n-1} \mathrm{D} + p_n) y = 0 \tag{2.2}$$

In brief,

$$\Phi(\mathrm{D}) y = 0$$

Extending VII to (2.2)

$$\begin{aligned} \Phi(\mathrm{D}) y &= (\mathrm{D}^n + p_1 \mathrm{D}^{n-1} + \ldots + p_{n-1} \mathrm{D} + p_n) y = \\ &= (\mathrm{D} - r_1)(\mathrm{D} - r_2) \ldots (\mathrm{D} - r_n) y = 0 \end{aligned} \tag{2.3}$$

where $r_1, \ldots, r_n$ are the roots of the auxiliary equation $\Phi(r) = r^n + p_1 r^{n-1} + \ldots + p_{n-1} r + p_n = 0$ and where this factoring in (2.3) is possible whenever $p_1, \ldots, p_n$ are constants and consequently $r_1, \ldots, r_n$ are too. This is called *operational factoring*.

1st Case. $r_1 \neq r_2 \neq \ldots \neq r_n$

The solution to $(\mathrm{D} - r_n) y = 0$ is the solution to (2.3). If y_n is such that $(\mathrm{D} - r_n) y_n = 0$ replacing in (2.3)

$$\Phi(\mathrm{D}) y_n = (\mathrm{D} - r_1)(\mathrm{D} - r_2) \ldots (\mathrm{D} - r_{n-1})(0) = 0$$

Solving $\dfrac{dy}{dx} - r_n y_n = 0 \qquad \dfrac{dy}{y_n} = r_n dx.$

Then $\ln y_n = r_n x + c_n = r_n x + \ln C_n$

And $y_n = e^{c_n} e^{r_n x} = C_n e^{r_n x}$ is a solution to (2.3).

Then by commutative property (VI of (1.1)) we have that

$$y_1 = C_1 e^{r_1 x} \quad ; y_2 = C_2 e^{r_2 x} \quad ; \ \ldots \ ; y_n = C_n e^{r_n x}$$

are all solutions to (2.3).

Then, by I of (1.1), the sum of two o more solutions is a solution of the homogeneous linear equation. That is,

$$y = C_1 e^{r_1 x} + \ldots + C_n e^{r_n x} \tag{2.4}$$

is a solution to (2.3).

2nd Case. Let's suppose $\Phi(r) = 0$ has a root of multiplicity P.

Then $\Phi(r)$ contains the factor $(r - r_1)^P$, that is

$$\Phi(r) = \Phi_1(r)(r - r_1)^P$$

where the polynomial $\Phi_1(r)$ is of degree $n - P$ with $\Phi_1(r_1) \neq 0$. Then our differential equation may be written as

$$\Phi(\mathrm{D}) = \Phi_1(\mathrm{D})(\mathrm{D} - r_1)^P y = 0 \tag{2.5}$$

And because of the former, $y = e^{r_1 x}$ is a solution to (2.5).

Now let $y = e^{r_1 x}\varphi(x)$. Using (1.7)

$$\Phi_1(\mathrm{D})(\mathrm{D} - r_1)^P (e^{r_1 x}\varphi(x)) = \Phi_1(\mathrm{D}) e^{r_1 x}\, \mathrm{D}^P(\varphi(x)) = 0$$

Then, as $e^{r_1 x} \neq 0$ we obtain $\mathrm{D}^P(\varphi(x)) = 0$. Note that $\varphi(x)$ is a polynomial of degree $P - 1$ and we obtain $\varphi(x) = C_1 + C_2 x + \ldots + C_p x^{p-1}$. Finally,

$$y = e^{r_1 x}\left(C_1 + C_2 x + \ldots + C_p x^{p-1}.\right) \tag{2.6}$$

That is, the part of the general solution that comes from the root r_1 of multiplicity P is given by (2.6). Then, the general solution will be given by the solution of $\Phi_1(\mathrm{D})y = 0$ added to (2.6), thus completing the n constants.

3rd Case. The auxiliary equation has complex and conjugate roots.

Let $r_1 = \alpha + \beta i$ and its conjugate $r_2 = \alpha - \beta i$.

That is, $\Phi(r) = \Phi_1(r)(r - r_1)(r - r_2)$.

The differential equation is $\Phi(\mathrm{D})y = \Phi_1(\mathrm{D})(\mathrm{D} - r_1)(\mathrm{D} - r_2)y = 0$.

Let

$$y = c_1 e^{(\alpha+\beta i)x} + c_2 e^{(\alpha-\beta i)x} \tag{2.7}$$

Since $e^{(\alpha+\beta i)x} = e^{\alpha}(\cos\beta + i\sin\beta)$, (2.7) becomes

$$\begin{aligned} y &= e^{\alpha x}\left[(c_1+c_2)\cos\beta x + i(c_1-c_2)\sin\beta x\right] \\ &= e^{\alpha x}\left[C_1\cos\beta x + C_2\sin\beta x\right] \end{aligned} \tag{2.8}$$

where C_1 and C_2 are arbitrary constants.

Let's discard that (2.8) satisfies the equation $\Phi(\mathrm{D})y = 0$:

$$\begin{aligned} \Phi(\mathrm{D})y &= \Phi_1(\mathrm{D})\,(\mathrm{D}-(\alpha+\beta i))\,(\mathrm{D}-(\alpha-\beta i))\,y = \\ &= \Phi_1(\mathrm{D})\,[(\mathrm{D}-\alpha)^2+\beta^2]\,y \end{aligned}$$

$$\Phi(\mathrm{D})\left[e^{\alpha x}(C_1\cos\beta x + C_2\sin\beta x)\right] = \Phi_1(\mathrm{D})\left[(\mathrm{D}-\alpha)^2+\beta^2\right]\left[e^{\alpha x}(C_1\cos\beta x + C_2\sin\beta x)\right]$$

By (1.8)

$$\Phi(\mathrm{D})y = \Phi_1(\mathrm{D})e^{\alpha x}(\mathrm{D}^2+\beta^2)(C_1\cos\beta x + C_2\sin\beta x)$$

Developing $(\mathrm{D}^2+\beta^2)(C_1\cos\beta x + C_2\sin\beta x) = 0$.

Then (2.8) is the solution to $\Phi(\mathrm{D})y = 0$.

3 LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

The ***general integral*** or ***general solution*** to a differential equation of higher order

$$\frac{d^n y}{dx^n} + p_1\frac{d^{n-1}y}{dx^{n-1}} + p_2\frac{d^{n-2}y}{dx^{n-2}} + \ldots + p_{n-1}\frac{dy}{dx} + p_n y = f(x) \tag{3.1}$$

in which $f(x)$ is a function of x and $p_1, \ldots, p_n$ are constants, is the one we obtain by adding the complementary solution (solution of $\Phi(\mathrm{D})y = 0$ as was seen in item 3) and any one particular integral from (3.1).

Observation 3.1. *Let us first note that we may write* $f(x) = f_1(x) + f_2(x)$ *and let's consider*

$$\Phi(\mathrm{D})y = f_1(x) \qquad \Phi(\mathrm{D})y = f_2(x) \tag{3.2}$$

We find two solutions y_{P_1} *and* y_{P_2} *solutions to* (3.2)*; that is,*

$$\Phi(\mathrm{D})y_{P_1} = f_1(x) \qquad \Phi(\mathrm{D})y_{P_2} = f_2(x)$$

and we obtain

$$\Phi(\mathrm{D})(y_{P_1} + y_{P_2}) = \Phi(\mathrm{D})y_{P_1} + \Phi(\mathrm{D})y_{P_2} = f_1(x) + f_2(x) = f(x).$$

In other words $y_{P_1} + y_{P_2}$ *is a particular solution of* (3.1)*, with which we conclude that by unfolding* $f(x)$ *we may apply some method to find* y_{P_1} *and maybe some other method to find* y_{P_2}*, since there are different methods for finding particular integrals.*

4 Particular integrals

In the analytic approach we dealt with the method of variation of constants and the method of undetermined coefficients to find particular integrals.

We will now see different procedures to find those integrals and which are called *operational methods.*

Let the differential equation be

$$\Phi(\mathrm{D})y = f(x) \tag{4.1}$$

The polynomial linear operator $\Phi(\mathrm{D})$ is given by

$$\Phi(\mathrm{D}) = \mathrm{D}^n + p_1 \mathrm{D}^{n-1} + p_2 \mathrm{D}^{n-2} + \ldots + p_n. \tag{4.2}$$

Clearing from (4.1)

$$y = \frac{1}{\Phi(\mathrm{D})} f(x) = \frac{1}{\mathrm{D}^n + p_1 \mathrm{D}^{n-1} + \ldots + p_n} f(x) \tag{4.3}$$

To interpret the nature of the operation let's start with $\mathrm{D}\, y = f(x)$, which implies that $y = \frac{1}{\mathrm{D}} f(x)$. Now, $\mathrm{D}\, y = \frac{dy}{dx} = f(x)$, that is $y = \int f(x)dx$, so it is natural to write

$$y = \frac{1}{\mathrm{D}} f(x) = \int f(x)dx \tag{4.4}$$

Analyzing

$$y = \frac{1}{\mathrm{D}^2} f(x) = \frac{1}{\mathrm{D}} \int f(x)dx$$

$$y = \iint f(x)dx^2$$

which means to integrate $f(x)$ twice successively.

Let us now consider

$$(\mathrm{D} - r)y = f(x) \quad r \text{ constant}, \tag{4.5}$$

$$y = \frac{1}{\mathrm{D} - r} f(x) \tag{4.6}$$

$$\frac{d(ye^{-rx})}{dx} = \frac{dy}{dx} e^{-rx} - rye^{-rx} = e^{-rx}\left(\frac{dy}{dx} - ry\right) = e^{-rx} f(x) \quad \text{by (4.5)}$$

Integrating

$$ye^{-rx} = \int e^{-rx} f(x)dx \Longrightarrow y = e^{rx} \int e^{-rx} f(x)dx$$

In other words, if $\frac{dy}{dx} - ry = f(x)$ it implies that $y = e^{rx} \int e^{-rx} f(x)dx$

Thus, from the latter

$$(\mathrm{D} - r)y = f(x) \qquad y = \frac{1}{\mathrm{D} - r} f(x) \qquad y = e^{rx} \int e^{-rx} f(x)dx \tag{4.7}$$

$$y = \frac{1}{\mathrm{D} - r} f(x) = e^{rx} \int e^{-rx} f(x)dx \tag{4.8}$$

Note that when $r = 0$, this equality is reduced to (4.4).

Let us now consider the different operational methods for solving (4.3) or solving the differential equation:

$$y = \frac{1}{\Phi(\mathrm{D})} f(x).$$

First, we try to find a particular integral.

Solve

$$\frac{dy}{dx} + Py = Q \tag{4.9}$$

in which P and Q are functions of x.

Solving $\quad \frac{dy}{dx} + Py = 0 \qquad y_H = Ce^{-\int P dx}$

A particular integral is given by $y_P = e^{-Px} \int e^{Pdx} Q dx$.

Then, the general solution is given by

$$y_G = e^{-\int P dx} \left[\int e^{Pdx} Q dx + C \right] \tag{4.10}$$

where (4.10) represents the solution to the linear equation (4.9).

5 Methods

Method 5.1 Successive Integrations. The method consists in solving a succession of differential equations of type (4.8), differential equations of first order.

$$\begin{aligned} y &= \frac{1}{\Phi(\mathrm{D})} f(x) = \frac{1}{(\mathrm{D} - r_1)(\mathrm{D} - r_2) \dots (\mathrm{D} - r_n)} f(x) = \\ &= e^{r_1 x} \int e^{-r_1 x} e^{r_2 x} \int e^{-r_2 x} e^{r_3 x} \int \dots e^{r_n x} \int e^{-r_n x} f(x) dx^n \end{aligned}$$

We may apply the method if some or all of the roots $r_1, \dots, r_n$ are equal.

Method 5.2 Decomposing operators into simple fractions. The successive integrations of the method above may complicate the search for a solution; so given

$$y = \frac{1}{\Phi(\mathrm{D})} f(x) = \frac{1}{(\mathrm{D} - r_1)(\mathrm{D} - r_2) \dots (\mathrm{D} - r_n)} f(x)$$

It would be better to break the operator into simple fractions:

$$y = \frac{1}{\Phi(\mathrm{D})} f(x) = \left[\frac{A_1}{(\mathrm{D} - r_1)} + \frac{A_2}{(\mathrm{D} - r_2)} + \ldots + \frac{A_n}{(\mathrm{D} - r_n)} \right] f(x)$$

with $A_1, \ldots, A_n$ being constants to be determined.

The operator within brackets is called *Heaviside Development* of the inverse operator $\frac{1}{\Phi(\mathrm{D})}$. Again, by (4.8) we have

$$y = \frac{1}{\Phi(\mathrm{D})} f(x) = A_1 e^{r_1 x} \int e^{-r_1 x} f(x) dx + A_2 e^{r_2 x} \int e^{-r_2 x} f(x) dx + \ldots + A_n e^{r_n x} \int e^{-r_n x} f(x) dx \quad (5.1)$$

which constitutes a series of simple integrals.

METHOD 5.3 OPERATOR TRANSLATIONS

By (1.6)

$$\Phi(\mathrm{D})(e^{rx}\varphi(x)) = e^{rx}\Phi(\mathrm{D} + r)\varphi(x) \quad (5.2)$$

We will demonstrate the same equivalence for the inverse operator.

$$\Phi(\mathrm{D}) e^{rx} \frac{1}{\Phi(\mathrm{D} + r)} \varphi(x) = e^{rx} \Phi(\mathrm{D} + r) \frac{1}{\Phi(\mathrm{D} + r)} \varphi(x) = e^{rx}\varphi(x)$$

$$\Phi(\mathrm{D}) \frac{1}{\Phi(\mathrm{D})} e^{rx}\varphi(x) = e^{rx}\varphi(x)$$

Equating the first members of the equations above we have

$$\frac{1}{\Phi(\mathrm{D})}(e^{rx}\varphi(x)) = e^{rx} \frac{1}{\Phi(\mathrm{D} + r)} \varphi(x) \quad (5.3)$$

Properties (5.2) and (5.3) are called *rules of exponential translation.*

They are used to move exponential functions outside the application of operators. Thus the name of *translation* and *inverse translation* of operators, names for (5.2) and (5.3), respectively.

Let's see by (1.8):

$$(\mathrm{D} - r)^k y = e^{rx}\, \mathrm{D}^k (e^{-rx} y) \quad (5.4)$$

$$e^{rx}\,\mathrm{D}^{-k}\left[e^{-rx}\,\underbrace{e^{rx}\,\mathrm{D}^{k}(e^{-rx}y)}\right] = 1y \tag{5.5}$$

Equation (5.5) is just the identity applied to y in the left-hand side of the equation, written as multiplications.

However

$$e^{rx}\,\mathrm{D}^{k}(e^{-rx}y) = (\mathrm{D}-r)^{k}y \qquad \text{by} \ \ (5.4)$$

Then, in (5.5)

$$e^{rx}\,\mathrm{D}^{-k}\left[e^{-rx}(\mathrm{D}-r)^{k}y\right] = 1y \tag{5.6}$$

Then, from (5.6), $\left[(\mathrm{D}-r)^{k}\right]^{-1}y = e^{rx}\,\mathrm{D}^{-k}(e^{-rx}y)$

Finally, we have $(\mathrm{D}-r)^{-k}y = e^{rx}\,\mathrm{D}^{-k}(e^{-rx}y)$ or

$$\frac{1}{(\mathrm{D}-r)^{k}}y = e^{rx}\frac{1}{\mathrm{D}^{k}}(e^{-rx}y) \tag{5.7}$$

METHOD 5.4 IF $f(x)$ IS IN THE FORM x^m

$$y = \frac{1}{\Phi(\mathrm{D})}x^m = (A_0 + A_1\,\mathrm{D} + A_2\,\mathrm{D}^2 + \ldots + A_n\,\mathrm{D}^n)x^m \quad \text{with } A_0 \neq 0. \tag{5.8}$$

The expression within brackets is obtained by developing $\dfrac{1}{\Phi(\mathrm{D})}$, according to increasing powers of D and eliminating all terms after D^n. They are deleted because $\mathrm{D}^n\,x^m = 0$ if $n > m$.

METHOD 5.5 IF $f(x) = e^{\alpha x}$ Let's see

$$\Phi(\mathrm{D})e^{\alpha x} = \sum_{i=0}^{n} p_i\,\mathrm{D}^{n-i}\,e^{\alpha x} = \sum_{i=0}^{n} p_i\alpha^{n-i}e^{\alpha x} = \Phi(\alpha)e^{\alpha x}$$

Multiplying the above by $\dfrac{1}{\Phi(\mathrm{D})}$,

$$\frac{1}{\Phi(\mathrm{D})}\Phi(\mathrm{D})e^{\alpha x} = \frac{\Phi(\alpha)}{\Phi(\mathrm{D})}e^{\alpha x}$$

That is,

$$\Phi(\alpha)\frac{1}{\Phi(\mathrm{D})}e^{\alpha x} = e^{\alpha x}$$

Finally,

$$\frac{1}{\Phi(\mathrm{D})}e^{\alpha x} = \frac{1}{\Phi(\alpha)}e^{\alpha x} \tag{5.9}$$

METHOD 5.6 IF $f(x) = x\varphi(x)$.

Let ψ be a function of x, that is, $\psi = \psi(x)$. And let a function, $u(x) = u$ such that $u(x) = x\psi(x)$ or $u = x\psi$.

$$\mathrm{D}\,u = x\,\mathrm{D}\,\psi + \psi$$

$$\mathrm{D}^n\,u = x\,\mathrm{D}^n\,\psi + \left(\frac{d\,\mathrm{D}^n}{d\,\mathrm{D}}\right)\psi$$

Then

$$\begin{aligned}\Phi(\mathrm{D})x\psi \;&= \sum_{i=0}^{n} p_i\,\mathrm{D}^{n-i}(x\psi) = \sum_{i=0}^{n} p_i x\,\mathrm{D}^{n-i}\,\psi + \sum_{i=0}^{n} p_i\left(\frac{d\,\mathrm{D}^{n-i}}{d\,\mathrm{D}}\right)\psi = \\ &= \sum_{i=0}^{n} p_i x\,\mathrm{D}^{n-i}\,\psi + \Phi'(\mathrm{D})\psi = x\Phi(\mathrm{D})\psi + \Phi'(\mathrm{D})\psi\end{aligned} \tag{5.10}$$

Now let

$$\varphi = \Phi(\mathrm{D})\psi \qquad\qquad \psi = \frac{1}{\Phi(\mathrm{D})}\varphi \tag{5.11}$$

Then, applying (5.10) on (5.11) we have

$$\Phi(\mathrm{D})x\frac{1}{\Phi(\mathrm{D})}\varphi = x\Phi(\mathrm{D})\frac{1}{\Phi(\mathrm{D})}\varphi + \Phi'(\mathrm{D})\frac{1}{\Phi(\mathrm{D})}\varphi$$

Clearing the first term of the right-hand side of the equation

$$x\varphi = \Phi(\mathrm{D})x\frac{1}{\Phi(\mathrm{D})}\varphi - \Phi'(\mathrm{D})\frac{1}{\Phi(\mathrm{D})}\varphi$$

Then, operating $\dfrac{1}{\Phi(\mathrm{D})}$ on both sides we have

$$\frac{1}{\Phi(\mathrm{D})}x\varphi = x\frac{1}{\Phi(\mathrm{D})}\varphi - \frac{\Phi'(\mathrm{D})}{[\Phi(\mathrm{D})]^2}\varphi \tag{5.12}$$

Then, by (5.12) and since $f(x) = x\varphi(x)$,

Pages 206 to 313 are not included in this book preview.

4 The Adams Method

We may apply this method whenever we have some initial values; that is, $x_1, x_2, x_3, \ldots, x_n$ and their corresponding values of $y_1, y_2, y_3, \ldots, y_n$. These can be found using the Taylor series or any other method studied before. The method consists of finding a value y_{n+1} for $x = x_{n+1} = x_1 + nh$.

If we have a differential equation of the form

$$y'(x) = f(x, y(x)) \tag{4.1}$$

with initial conditions $y = y_0$ when $x = x_0$, and we know the values $y_1, y_2, y_3, \ldots, y_n$, through the Adams Method we can determine the value of y_{n+1}, by proceeding in the following manner:

Integrating within the interval (x_n, x_{n+1}):

$$\int_{x_n}^{x_{n+1}} y'(x)dx = y(x_{n+1}) - y(x_n) \tag{4.2}$$

From (4.2) and since

$$h = x_{n+1} - x_n \tag{4.3}$$

$$y(x_{n+1}) = y(x_n) + \int_{x_n}^{x_n+h} f(x, y(x))dx \tag{4.4}$$

Making a change of variable

$$\begin{aligned} x &= x_n + ht \\ dx &= h\ dt \end{aligned} \tag{4.5}$$

Replacing (4.5) in (4.4) and changing the limits of integration we have

$$y(x_{n+1}) = y(x_n) + h \int_0^1 f\left[x_n + ht, y(x_n + ht)\right] dt \tag{4.6}$$

We now have to determine the expression inside the integral. Let's try to find it in the form of a polynomial $P(x)$, because it is easier to integrate.

Using subtractions

$$\Delta f_{n-1} = f_n - f_{n-1}$$

$$\Delta^2 f_{n-2} = \Delta f_{n-1} - \Delta f_{n-2} = f_n - 2f_{n-1} + f_{n-2}$$

$$\Delta^3 f_{n-3} = \Delta^2 f_{n-2} - \Delta^2 f_{n-3} \\ \vdots \tag{4.7}$$

If we make

$$P(x) = f_n + \frac{\Delta f_{n-1}}{h}\frac{(x - x_n)}{1!} + \frac{\Delta^2 f_{n-2}}{h^2}\frac{(x - x_n)(x - x_{n-1})}{2!} + \dots \tag{4.8}$$

From this expression it may be observed that if

$$\begin{array}{ll} x = x_n & P(x) = f_n \\ x = x_{n-1} & P(x) = f_n + \dfrac{\Delta f_{n-1}}{h}\dfrac{x_{n-1} - x_n}{1!} = f_n - f_n + f_{n-1} = f_{n-1} \\ x = x_{n-2} & P(x) = f_n - 2f_n + 2f_{n-1} + f_n - 2f_{n-1} + f_{n-2} = f_{n-2} \\ \vdots & \vdots \end{array} \tag{4.9}$$

Thus we arrive to the fact that for $x_n, x_{n-1}, x_{n-2}, \dots, x_{n-m}$, the polynomial takes the values $f_n, f_{n-1}, f_{n-2}, \dots, f_{n-m}$, respectively; which implies that $P(x)$ is the polynomial we were looking for.

From expression (4.5)

$$t = \frac{x - x_n}{h}\ ;\ \frac{x - x_{n-1}}{h} = \frac{x - x_n + h}{h} = t + 1\ ;\ \dots \tag{4.10}$$

Later, replacing (4.10) in (4.8)

$$P(t) = f_n + \Delta f_{n-1}\frac{t}{1!} + \Delta^2 f_{n-2}\frac{t(t+1)}{2!} + \dots \tag{4.11}$$

Then substituting in (4.6)

$$y(x_{n+1}) = y(x_n) + h\int_0^1 \left[f_n + \Delta f_{n-1}\frac{t}{1!} + \Delta^2 f_{n-2}\frac{t(t+1)}{2!} + \dots\right] dt \tag{4.12}$$

Integrating

$$y(x_{n+1}) = y(x_n) + h\left(f_n + \frac{1}{2}\Delta f_{n-1} + \frac{5}{12}\Delta^2 f_{n-2} + \frac{3}{8}\Delta^3 f_{n-3} + \frac{251}{720}\Delta^4 f_{n-4} + \dots\right) \tag{4.13}$$

For example, if values x_0, x_1, x_2, x_3, x_4 and their respective y_0, y_1, y_2, y_3, y_4 are given as data we can diagram a table which will allow us to compute $y_5, y_6, y_7, \ldots$.

$$\begin{array}{cccccccc}
 & x_0 & y_0 & f_0 & & & & \\
 & & & & \Delta f_0 & & & \\
 & x_1 & y_1 & f_1 & & \Delta^2 f_0 & & \\
 & & & & \Delta f_1 & & \Delta^3 f_0 & \\
\textit{Data} & x_2 & y_2 & f_2 & & \Delta^2 f_1 & & \Delta^4 f_0 \\
 & & & & \Delta f_2 & & \Delta^3 f_1 & \\
 & x_3 & y_3 & f_3 & & \Delta^2 f_2 & & \Delta^4 f_1 \\
 & & & & \Delta f_3 & & \Delta^3 f_2 & \\
 & x_4 & y_4 & f_4 & & \Delta^2 f_3 & & \\
 & & & & \Delta f_4 & & & \\
 & x_5 & y_5 & f_5 & & & & \\
 & & & & & & & \\
 & x_6 & y_6 & & & & &
\end{array} \tag{4.14}$$

Below is an example to illustrate the method.

***Example* 4.1.** Given the differential equation

$$y' = y + x \tag{a}$$

find the values from $x_0 = 0$, $y_0 = 0$ to $x = 0.4$ with $h = 0.1$, using the Taylor series.

For $x_0 = 0$, $x_1 = 0.1$, $x_2 = 0.2$ we are given the values of y_0, y_1, y_2, respectively. Applying the Adams Method establish the values for y_3, y_4.

For $x_0 = 0$, $y_0 = 0$

$$x_1 = 0.1 \qquad y_1 = \frac{(0.1)^2}{2!} + \frac{(0.1)^3}{3!} + \frac{(0.1)^4}{4!} + \ldots$$

$$y_1 = 0.005 + 0.000166 + 0.000004 + \ldots = 0.005170$$

$$x_2 = 0.2 \qquad y_2 = \frac{(0.2)^2}{2!} + \frac{(0.2)^3}{3!} + \frac{(0.2)^4}{4!} + \ldots$$

$$y_2 = 0.02 + 0.001333 + 0.000066 + \ldots = 0.021399$$

With these data we can determine the table to compute the values of y_3, y_4.

	x_i	y_i	y'_i	Δf_i	Δf_i^2
	0	0	0		
				0.105170	
Data	0.1	0.005170	0.105170		0.011059
				0.116229	
	0.2	0.021399	0.221399		0.012183
				0.128412	
	0.3	0.049811	0.349811		
	0.4	0.091720			

Applying formula (4.13)

$$\begin{aligned} y_3 &\cong 0.021399 + 0.1\left(0.221399 + \frac{1}{2}0.116229 + \frac{5}{12}0.011059\right) = \\ &= 0.021399 + 0.1\,(0.221399 + 0.058114 + 0.004607) = 0.049811 \end{aligned}$$

Then,

$$\begin{aligned} y_4 &\cong 0.049811 + 0.1\left(0.349811 + \frac{1}{2}0.128412 + \frac{5}{12}0.012183\right) = \\ &= 0.049811 + 0.1\,(0.349811 + 0.064206 + 0.005076) = 0.091720 \end{aligned}$$

We could compute another column and determine y_5 and successively $y_6, y_7, \ldots$.

5 THE RUNGE-KUTTA METHOD

If we have a differential equation of the form

$$y' = f(x, y) \tag{5.1}$$

with the initial conditions $y = y_0$ when $x = x_0$.

It can be written as

$$y(x) = y_0 + \int_{x_0}^{x} f(x, y(x))dx \tag{5.2}$$

By using the Simpson Rule, (5.2) can be expressed as

$$y_{n+1} = y_n + (k_1 + 2k_2 + 2k_3 + k_4)/6 \tag{5.3}$$

$$\begin{aligned} k_1 &= hf(x_n, y_n) \\ k_2 &= hf(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1) \\ k_3 &= hf(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_2) \\ k_4 &= hf(x_n + h, y_n + k_3) \end{aligned} \tag{5.4}$$

***Example* 5.1.** Determine, using the Runge-Kutta method, the values for y_1, y_2 of the differential equation

$$y' = y + x$$

with the conditions $y_0 = 0$ when $x_0 = 0$ for an interval $h = 0.1$.

Drawing a table like before:

x	Y	y'	k_1	k_2	k_3	k_4
0	0	0	0	0.005	0.00525	0.010525
0.1	0.00517	0.10517	0.010517	0.016042	0.016319	0.022148
0.2	0.02140					

The proof can be seen in "Collatz, Numerische Behandlung von Differentialgleichungen".

Chapter 4

APPLICATIONS APPROACH

1 Introduction

The reason for learning to solve ordinary differential equations through applications is based on the reciprocity between these equations and their applications. Since once we have the structure of the corresponding equation, we may attain operating skills by its computation.

We will define these differential equations as a mathematical model isomorphic to the real problem and approach them knowing that many of the problems appearing in different fields of Engineering and Physics are solved through linear ordinary differential equations with variable constants, e.g. the special equations of Hermite, Legendre, Bessel.

Eventhough the Bessel function was obtained to solve a differential equation associated to planet movements, it appears in many problems of electronics engineering, hydrodynamics, thermodynamics.

2 The Hermite Equation

This equation is of the form

$$\frac{d^2y}{dx^2} - x\frac{dy}{dx} + my = 0 \qquad\qquad m > 0 \text{ integer} \tag{2.1}$$

We will find the solution by using power series,

$$
\begin{aligned}
y &= \sum_{k=0}^{\infty} a_k x^k \\
y' &= \sum_{k=0}^{\infty} k a_k x^{k-1} \\
y'' &= \sum_{k=0}^{\infty} k(k-1) a_k x^{k-2}
\end{aligned}
\tag{2.2}
$$

Substituting (2.2) in (2.1) we have

$$
\sum_{k=0}^{\infty} k(k-1)a_k x^{k-2} - \sum_{k=0}^{\infty} k a_k x^k + m \sum_{k=0}^{\infty} a_k x^k = 0 \tag{2.3}
$$

or, the same,

$$
\sum_{k=0}^{\infty}(k+1)(k+2)a_{k+2}x^k - \sum_{k=0}^{\infty} k a_k x^k + m \sum_{k=0}^{\infty} a_k x^k = 0 \tag{2.4}
$$

$$
\sum_{k=0}^{\infty} \left[(k+1)(k+2)a_{k+2} - k a_k + m a_k\right] x^k = 0 \tag{2.5}
$$

In the latter expression we must cancel

$$
(k+1)(k+2)a_{k+2} = (k-m)a_k \tag{2.6}
$$

$$
a_{k+2} = \frac{k-m}{(k+2)(k+1)} a_k \tag{2.7}
$$

From (2.7) we obtain the following values

$$
\begin{aligned}
a_2 &= -\frac{m}{2\times 1} a_0 \\
a_3 &= \frac{1-m}{3\times 2} a_1 \\
a_4 &= \frac{2-m}{4\times 3} a_2 = \frac{-m\,(2-m)}{4\times 3\times 2\times 1} a_0 \\
a_5 &= \frac{3-m}{5\times 4} a_3 = \frac{(1-m)(3-m)}{5\times 4\times 3\times 2\times 1} a_1 \\
a_6 &= \frac{4-m}{6\times 5} a_4 = \frac{-m(2-m)(4-m)}{6\times 5\times 4\times 3\times 2\times 1} a_0 \\
a_7 &= \frac{5-m}{7\times 6} a_5 = \frac{(5-m)(3-m)(1-m)}{7\times 6\times 5\times 4\times 3\times 2\times 1} a_1
\end{aligned}
\tag{2.8}
$$

Finally, the general solution to the differential equation is given by

$$\begin{aligned} y = a_0 &\left[1 - \frac{m}{2!}x^2 - \frac{m(2-m)}{4!}x^4 - \frac{m(2-m)(4-m)}{6!}x^6 - \dots\right] + \\ +a_1 &\left[x + \frac{1-m}{3!}x^3 - \frac{(1-m)(3-m)}{5!}x^5 + \frac{(1-m)(3-m)(5-m)}{7!}x^7 + \dots\right] \end{aligned} \tag{2.9}$$

As may be observed $a_{m+2} = a_{m+4} = a_{m+6} = 0$ and if we make $k = m-2;\ m-4;\ m-6;\ \dots$, we have

$$a_{m-2} = \frac{-m(m-1)}{2}a_m = -\binom{m}{2}a_m$$

$$a_{m-4} = \frac{m(m-1)(m-2)(m-3)}{4\times 2}a_m = 1\times 3\binom{m}{4}a_m$$

And so on, successively. If we make $a_m = 1$ and $a_{m-1} = 0$ we will have a particular integral called the ***Hermite polynomial***, which is in the form:

$$He_m(x) = x^m - \binom{m}{2}x^{m-2} + 1\times 3\binom{m}{4}x^{m-4} - \dots = (-1)^n e^{\frac{x^2}{2}} \frac{d^n\left(e^{-\frac{x^2}{2}}\right)}{dx^n} \tag{2.10}$$

3 The Legendre Equation

This kind of differential equation is given in the form

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \tag{3.1}$$

or also

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + n(n+1)y = 0 \tag{3.2}$$

where n equals $0, 1, 2, 3, \dots$

We may find the solution through the serial expansion of

$$y = \sum_{k=0}^{\infty} a_k x^k \tag{3.3}$$

Differentiating we have

$$y' = \sum_{k=1}^{\infty} k a_k x^{k-1} \qquad (3.4) \qquad\qquad y'' = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} \qquad (3.5)$$

From there

$$2xy' = \sum_{k=1}^{\infty} 2k a_k x^k = \sum_{k=0}^{\infty} 2k a_k x^k \tag{3.6}$$

$$\begin{aligned} (1-x^2)y'' &= \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} - \sum_{k=2}^{\infty} k(k-1) a_k x^k = \\ &= \sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k - \sum_{k=0}^{\infty} k(k-1) a_k x^k = \\ &= \sum_{k=0}^{\infty} \left[(k+2)(k+1) a_{k+2} - k(k-1) a_k\right] x^k \end{aligned} \tag{3.7}$$

Substituting (3.3), (3.6) and (3.7) in (3.1) we obtain

$$\sum_{k=0}^{\infty} \left[(k+2)(k+1) a_{k+2} - k(k-1) a_k - 2k a_k + n(n+1) a_k\right] x^k = 0 \tag{3.8}$$

That is, the following must be valid

$$(k+2)(k+1) a_{k+2} - k(k-1) a_k - 2k a_k + n(n+1) a_k = 0 \tag{3.9}$$

or, the same,

$$\begin{aligned} &(k+2)(k+1) a_{k+2} + \left[n(n+1) - 2k - k(k-1)\right] a_k = 0 \\ &(k+2)(k+1) a_{k+2} - (k-n)(k+1+n) a_k = 0 \end{aligned} \tag{3.10}$$

Clearing

$$a_{k+2} = -\frac{(n-k)(k+1+n)}{(k+2)(k+1)} a_k \qquad k \geq 0 \tag{3.11}$$

Then, making $k = 0, 1, 2, 3, \ldots$

$$\begin{aligned} a_2 &= -\tfrac{n(n+1)}{2!}a_0 & a_3 &= -\tfrac{(n-1)(n+2)}{3!}a_1 \\ a_4 &= -\tfrac{n(n-2)(n+1)(n+3)}{4!}a_0 & a_5 &= -\tfrac{(n-1)(n-3)(n+2)(n+4)}{5!}a_1 \\ a_6 &= -\tfrac{n(n-2)(n-4)(n+1)(n+3)(n+5)}{6!}a_0 & a_7 &= -\tfrac{(n-1)(n-3)(n-5)(n+2)(n+4)(n+6)}{7!}a_1 \end{aligned} \tag{3.12}$$

Substituting the values of (3.12) in (3.3) gives

$$y_1 = a_0\left[1 - \tfrac{n(n+1)}{2!}x^2 - \tfrac{n(n-2)(n+1)(n+3)}{4!}x^4 - \tfrac{n(n-2)(n-4)(n+1)(n+3)(n+5)}{6!}x^6 + \ldots\right] \tag{3.13}$$

$$y_2 = a_1\left[x - \tfrac{(n-1)(n+2)}{3!}x^3 + \tfrac{(n-1)(n-3)(n+2)(n+4)}{5!}x^5 - \tfrac{(n-1)(n-3)(n-5)(n+2)(n+4)(n+6)}{7!}x^7 + \ldots\right] \tag{3.14}$$

If we put $y_1 + y_2$ we obtain the general solution to (3.1).

According to the previous rule, we have that the last two series converge for $|x| < 1$.

As we can observe in (3.13) and (3.14), if $n = 0$ or is an even positive integer, the expression in (3.13) becomes a polynomial of n degree, $P_n(x)$ and the expression in (3.14), an infinite series $Q_n(x)$.

If instead n is an odd positive integer, the expression in (3.13) becomes an infinite series, whereas the expression in (3.14) remains as a polynomial of n degree, $P_n(x)$.

This means that the general solution to Legendre will be given in the form $y = a_0P_n(x) + a_1Q_n(x)$ or $y = a_0Q_n(x) + a_1P_n(x)$, given $n = 0, 2, 4, \ldots$ or $n = 1, 3, 5, 7, \ldots$, respectively.

In other words, the general solution to the Legendre equation contains a solution of polynomials and an infinite solution.

The functions of the second type $Q_n(x)$ are of relative importance in applied mathematics.

Therefore, from the expression in (3.11) we will try to find the polynomials for any value of n, either zero, even or odd.

$$a_{k+2} = -\frac{(n-k)(k+1+n)}{(k+2)(k+1)}a_k$$

By making $k = n-2, n-4, \ldots$

$$\begin{aligned} a_{n-2} &= -\frac{n(n-1)}{2(2n-1)}a_n \\ a_{n-4} &= -\frac{(n-2)(n-3)}{4(2n-3)}a_{n-2} = \frac{n(n-1)(n-2)(n-3)}{2{\times}4(2n-1)(2n-3)}a_n \end{aligned} \tag{3.15}$$

Therefore,

$$y = a_n\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2{\times}4(2n-1)(2n-3)}x^{n-4} - \ldots\right] \tag{3.16}$$

If we substitute a_n in a way that $y(1) = 1$ for any value of n, then we have

$$a_n = \frac{1{\times}3{\times}5{\times}7{\times}\ldots(2n-1)}{n!} = \frac{(2n)!}{2^n(n!)^2} \tag{3.17}$$

where

$$P_n(x) = \frac{(2n)!}{2^n(n!)^2}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2{\times}4(2n-1)(2n-3)}x^{n-4} - \ldots\right] \tag{3.18}$$

The expression in (3.18) is called the ***Legendre polynomial*** or function of the first type. From this latter expression we have different polynomials for different values of n.

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= \frac{1}{2}(3x^2-1) \\ P_3(x) &= \frac{1}{2}(5x^3-3x) \\ P_4(x) &= \frac{1}{8}(35x^4-30x^2+3) \\ P_5(x) &= \frac{1}{8}(63x^5-70x^3+15x) \end{aligned} \tag{3.19}$$

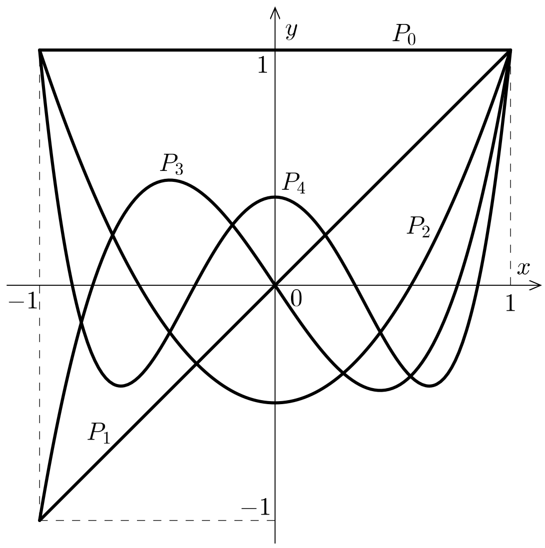

Figure 4.1

If we substitute $x = 1$, each of the polynomials will be equal to one, as we can see in (3.19), since we have established $y(1) = 1$.

These polynomials are represented in Figure 4.1.

Then

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x$$

As we can see, if $n \neq 0, 1, 2, 3$, we have two independent solutions:

$$y_1 = J_n(x) \text{ and } y_2 = J_{-n}(x) \text{ for } n > 0 \text{ fraction}$$

The general solution to (6.1) is

$$y = AJ_n(x) + BJ_{-n}(x) \tag{6.37}$$

Instead if n is a non negative integer, we have the particular integral J_n.

To obtain the general solution we must use the Bessel functions of the second type, which we will not include in this study.

7 The Thomson functions or the Ber–Bei functions

These functions are solutions to a special Bessel differential equation (of great importance in electricity problems) given in the form

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} - ixy = 0 \tag{7.1}$$

We had already obtained a Bessel differential equation of type

$$z^2\frac{d^2y}{dx^2} + z\frac{dy}{dx} + (z^2 - n^2)y = 0 \tag{7.2}$$

Its solution was

$$y = J_n(z) \tag{7.3}$$

If we make $z = ax$ in the expression in (7.2) we obtain

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (a^2x^2 - n^2)y = 0 \tag{7.4}$$

And its solution will be

$$y = J_n(ax) \tag{7.5}$$

Then, for $n = 0$

$$\begin{aligned} x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + a^2x^2y = 0 \\ x\frac{d^2y}{dx^2} + \frac{dy}{dx} + a^2xy = 0 \end{aligned} \tag{7.6}$$

where the general solution will be given by

$$y = J_0(ax) \tag{7.7}$$

As we can see, the expression in (7.6) is equal to the solution in (7.1), where $a^2 = -i$; that is, the solution to (7.1) is given in the form

$$y = J_0\left((-i)^{\frac{1}{2}}x\right) \tag{7.8}$$

and since

$$(-i)^{\frac{1}{2}} = i^{\frac{3}{2}} \tag{7.9}$$

(7.8) is reduced to

$$y = J_0\left(i^{\frac{3}{2}}x\right) \tag{7.10}$$

Then, substituting in the expression in (6.25) we have

$$\begin{aligned} J_0\left(i^{\frac{3}{2}}x\right) &= 1 - \frac{i^3x^2}{2^2} + \frac{i^6x^4}{2^4(2!)^2} - \frac{i^9x^6}{2^6(3!)^2} + \frac{i^{12}x^8}{2^8(4!)^2} - \frac{i^{15}x^{10}}{2^{10}(5!)^2} + \ldots = \\ &= 1 + \frac{ix^2}{2^2} - \frac{x^4}{2^4(2!)^2} - \frac{ix^6}{2^6(3!)^2} + \frac{x^8}{2^8(4!)^2} + \frac{ix^{10}}{2^{10}(5!)^2} - \ldots = \\ &= \left[1 - \frac{x^4}{2^24^2} + \frac{x^8}{2^24^26^28^2} - \ldots\right] + i\left[\frac{x^2}{2^2} - \frac{x^6}{2^24^26^2} + \frac{x^{10}}{2^24^26^28^210^2} - \ldots\right] = \\ &= \left[1 + \sum_{k=1}^{\infty}\frac{(-1)^kx^{4k}}{2^24^26^28^2\ldots(4k)^2}\right] + i\left[-\sum_{k=1}^{\infty}(-1)^k\frac{x^{4k-2}}{2^24^26^28^2\ldots(4k-2)^2}\right] \end{aligned} \tag{7.11}$$

Then, if we name

$$\begin{aligned} \mathrm{ber}(x) &= 1 + \sum_{k=1}^{\infty}\frac{(-1)^kx^{4k}}{2^24^26^28^2\ldots(4k)^2} \\ \mathrm{bei}(x) &= -\sum_{k=1}^{\infty}(-1)^k\frac{x^{4k-2}}{2^24^26^28^2\ldots(4k-2)^2} \end{aligned} \tag{7.12}$$

we arrive at

$$J_0\left(i^{\frac{3}{2}}x\right) = \mathrm{ber}(x) + i\ \mathrm{bei}(x) \tag{7.13}$$

which is the solution to (7.1).

The graphs of these functions can be seen in Figure 4.3.

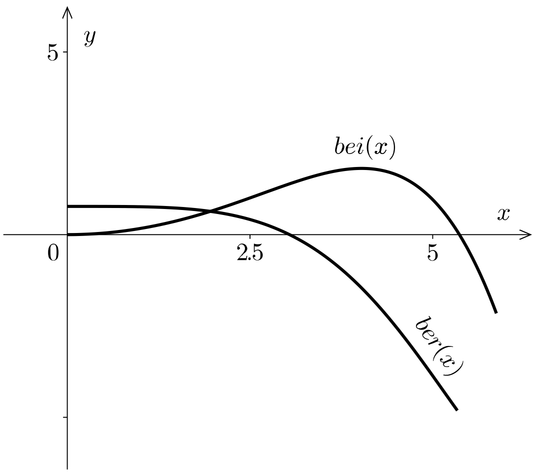

Figure 4.3

8 Modeling applications

Modeling is the process of reconstruction of a real system to a form called model, which may be analyzed.

We will introduce the concept by using a simple example.

***Example* 8.1.** We have a closed electrical circuit formed by a resistance of R ohms, an inductance of L henries, a condenser of capacitance C and an electromotive force $E(t)$.

Let x be an input reading in volts and an output reading in amperes.

Let the differential equation be

$$a\frac{d^2z}{dt^2} + b\frac{dz}{dt} + cz = w(t)$$

The changes produced by $w(t)$ maintain the variation of z in time; w constitutes the input of the equation and z, the output. If now a, b and c are assigned values related to R, L and C in a convenient manner, so that the relation between w and z is identical to the one between x and y, we say that the systems are isomorphic. In other words, we have modelized the real system.

The corresponding equation is

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} = E'(t)$$

which represents the mathematical model isomorphic to the electrical circuit.

The practical value of isomorphism is that we can predict how the real system is going to behave under certain conditions.

It may occur that it is too difficult to expose the real system to a direct trial. That may be because it is not easily accessible or it has not been built or is not yet available.

Summarizing, a mathematical model or isomorphism is a set of equations based on a description of a real system and created with the purpose of predicting behavior related to the real behavior, be it a system of science or engineering.

This page intentionally left blank

Copyrighted Material

GUIDED PRACTICE

***Example* 1.**

a) Solve:

$$t\frac{d^2y}{dt^2} + \frac{dy}{dt} + ty = 0 \tag{1.1}$$

such that $y(t) = 1$, $y'(t) = 0$ when $t = 0$.

Solution: On first inspection we see this is a linear differential equation, where the coefficients are non constant polynomials. Soon we associate the solving method to one of the methodological approach: the Laplace Transform. We can also see that the exercise is a first order Bessel equation and that we can take the applications approach.

In order to illustrate the general method, we will choose the first one.

Considering the properties of the operators for $t = 0$ we have

$$\mathcal{L}\{ty''\} = -\frac{d}{dp}\left[p^2Y(p) - p\right] = -p^2Y'(p) - 2pY(p) + 1$$

$$\mathcal{L}\{ty\} = -\frac{dY(p)}{dp} = -Y'(p)$$

$$\mathcal{L}\{y'\} = pY(p) - 1$$

Then, the equation in (1.1) transformed is

$$-p^2Y'(p) - 2pY(p) + 1 + pY(p) - 1 - Y'(p) = 0$$

$$(p^2+1)\frac{dY(p)}{dp} + pY(p) = 0$$

Separating variables

$$\frac{dY(p)}{Y(p)} + \frac{p\,dp}{p^2+1} = 0$$

$$\ln Y(p) + \frac{1}{2}\ln(p^2+1) = \ln c$$

If $k+1=n$ we have

$$J_{m+1}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(n-1)!\Gamma(m+n+1)}\left(\frac{x}{2}\right)^{m+2n-1}$$

From 5

$$\begin{aligned}
J_{m-1}= &\sum_{n=0}^{\infty}\frac{(-1)^{n}}{n!\Gamma(m+n)}\left(\frac{x}{2}\right)^{m+2n-1}\\
J_{m-1}+J_{m+1}= &\left(\frac{x}{2}\right)^{m-1}\frac{1}{\Gamma(m)}+\sum_{n=1}^{\infty}\frac{(-1)^{n}}{n!\Gamma(m+n)}\left(\frac{x}{2}\right)^{m+2n-1}+\\
&-\sum_{n=1}^{\infty}\frac{(-1)^{n}}{(n-1)!\Gamma(m+n+1)}\left(\frac{x}{2}\right)^{m+2n-1}=\\
&=\left(\frac{x}{2}\right)^{m-1}\left[\frac{1}{\Gamma(m)}+\sum_{n=1}^{\infty}\left(\frac{(-1)^{n}}{n!\Gamma(m+n)}\left(\frac{x}{2}\right)^{2n}-\frac{(-1)^{n}}{(n-1)!\Gamma(m+n+1)}\left(\frac{x}{2}\right)^{2n}\right)\right]=\\
&=\left(\frac{x}{2}\right)^{m-1}\left[\frac{1}{\Gamma(m)}+\sum_{n=1}^{\infty}(-1)^{n}\left(\frac{m+n}{n!\Gamma(m+n+1)}-\frac{n}{n!\Gamma(m+n+1)}\right)\left(\frac{x}{2}\right)^{2n}\right]=\\
&=\left(\frac{x}{2}\right)^{m-1}\left[\frac{1}{\Gamma(m)}+\sum_{n=1}^{\infty}(-1)^{n}\frac{m}{n!\Gamma(m+n+1)}\left(\frac{x}{2}\right)^{2n}\right]=\\
&=\left(\frac{x}{2}\right)^{m-1}m\left[\frac{1}{m\Gamma(m)}+\sum_{n=1}^{\infty}(-1)^{n}\frac{1}{n!\Gamma(m+n+1)}\left(\frac{x}{2}\right)^{2n}\right]=\\
&=m\sum_{n=0}^{\infty}\frac{(-1)^{n}}{n!\Gamma(m+n+1)}\left(\frac{x}{2}\right)^{m+2n-1}=\\
&=\frac{2m}{x}J_m(x)
\end{aligned}$$

***Example* 6.** Find the solution to the following integral

$$f(t)=\int_0^{\infty}\frac{\sin^2 tz}{z^2}dz \tag{6.1}$$

By trigonometric equality we may express (6.1) as

$$f(t)=\int_0^{\infty}\left[\frac{1}{2z^2}-\frac{\cos(2tz)}{2z^2}\right]dz$$

Applying the Laplace Transform and inverting variables we obtain

$$\begin{aligned}
\mathcal{L}\left\{f(t)\right\} &= \int_0^\infty \left[\frac{1}{2pz^2} - \frac{p}{2z^2(p^2+4z^2)}\right] dz = \\
&= \int_0^\infty \frac{4z^2}{2p(p^2+4z^2)z^2} dz = \int_0^\infty \frac{2}{p(p^2+4z^2)} dz = \\
&= \frac{1}{p^2} \arctan \frac{2z}{p}\bigg|_0^\infty = \frac{\pi}{2p^2}
\end{aligned}$$

$$\mathcal{L}\left\{f(t)\right\} = \frac{\pi}{2p^2} \tag{6.2}$$

By property $\mathcal{L}\left\{t^n\right\} = \dfrac{n!}{p^{n+1}}$, being n a positive integer and by applying $\mathcal{L}^{-1}$ to (6.2) we have

$$f(t) = \frac{\pi}{2}t$$

Volver a MATHurgent